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Amadeu ALMEIDA COCO

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Robust Covering Problems: Formulations, Algorithms and an Application

JURY

M. S. ALBERTO URRUTIA	PROFESSOR ADJUNTO	Président
M. C. DA CRUZ CARNEIRO RIBEIRO	PROFESSOR - HDR	Rapporteur
M. C. DUHAMEL	MAITRE DE CONFERENCES	Examinateur
M. T. FERREIRA DE NORONHA	PROFESSOR ADJUNTO	Directeur de thèse
M. P. MICHELON	PROFESSEUR DES UNIVERSITES	Rapporteur
M. C. PRINS	PROFESSEUR DES UNIVERSITES	Examinateur
Mme A. C. SANTOS	MAITRE DE CONFERENCES - HDR	Directrice de thèse

AMADEU ALMEIDA COCO

ROBUST COVERING PROBLEMS: FORMULATIONS, ALGORITHMS AND APPLICATION

Thesis presented to the Graduate Program in Computer Science Department of the Federal University of Minas Gerais and to the Technological University of Troyes in partial fulfillment of the requirements for the degree of Doctor, respectively, in Computer Science Department and in the Laboratory of Industrial Systems Optimization.

Advisors: Andréa Cynthia Santos; Thiago Ferreira de Noronha;

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I dedicate this thesis to everybody who helped me during these four years.

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"I only know that I know nothing."
(Socrates)

Resumo

Dois problemas NP-Difíceis de otimização robusta foram estudados nesta tese: o Problema *min-max regret* de Cobertura de Conjuntos Ponderado (*min-max regret* WSCP, do inglês *min-max regret Weighted Set Covering Problem*) e o Problema *min-max regret* de Cobertura e Localização Maximal (*min-max regret* MCLP, do inglês *min-max regret Maximal Coverage Location Problem*). O *min-max regret* WSCP e o *min-max regret* MCLP são respectivamente, versões de otimização robusta do Problema de Cobertura de Conjuntos Ponderado e do Problema de Cobertura e Localização Maximal. Em ambos os problemas, o conjunto de parâmetros incertos é modelado por intervalos onde apenas os valores mínimo e máximo são conhecidos. Além disso, uma aplicação do *min-max regret* MCLP em logística pós-desastres é investigada nesta tese e, para esta aplicação, dois outros critérios de otimização robusta foram derivados a partir do *min-max* MCLP: o *max-max* MCLP e o *max-min* MCLP. Em termos de métodos, quatro formulações matemáticas, três algoritmos exatos e cinco heurísticas foram desenvolvidos e aplicados para ambos os problemas. Experimentos computacionais mostraram que os algoritmos exatos resolveram 14 de 75 instancias geradas para o *min-max regret* WSCP e todas as instâncias realísticas criadas para o *min-max regret* MCLP. Além disso, em quase todas as instâncias que não foram resolvidas na otimialidade, as heurísticas propostas nesta tese encontraram soluções tão boas quanto ou melhores que aquelas retornadas por meio dos algoritmos exatos. Quanto à aplicação em logística pós-desastres, os três modelos de otimização robusta (*max-max* MCLP, *max-min* MCLP and *min-max regret* MCLP) encontraram soluções similares para os cenários realísticos gerados a partir dos dados dos terremotos que atingiram Catmandu, Nepal em 2015.

Keywords: Otimização Combinatória, Meta-heurísticas, Algoritmos, Pesquisa Operacional, Logística.

Abstract

Two robust optimization NP-Hard problems are studied in this thesis: the *min-max regret* Weighted Set Covering Problem (*min-max regret* WSCP) and the *min-max regret* Maximal Coverage Location Problem (*min-max regret* MCLP). The *min-max regret* WSCP and *min-max regret* MCLP are, respectively, the robust optimization counterparts of the Set Covering Problem and of the Maximal Coverage Location Problem. The uncertain data in these problems is modeled by intervals and only the minimum and maximum values for each interval are known. However, while the *min-max regret* WSCP is still a theoretical problem, the *min-max regret* MCLP has an application in disaster logistics which is also investigated in this thesis. We have derived for the MCLP, two other robust optimization criteria: the *max-min lower scenario* MCLP and the *max-min upper scenario* MCLP. In terms of methods, four mathematical formulations, three exact algorithms and five heuristics were developed and applied to both problems. Computational experiments showed that the exact algorithms efficiently solved 14 out of 75 instances generated to the *min-max regret* WSCP and all realistic instances created to the *min-max regret* MCLP. For the simulated instances that was not solved to optimally in both problems, the heuristics developed in this thesis found solutions, as good as, or better than the best exact algorithm in almost all instances. Concerning the application in disaster logistics, the three robust models *max-min lower scenario* MCLP, *max-min upper scenario* MCLP and *min-max regret* MCLP found similar solutions for realistic scenarios of the earthquakes that hit Kathmandu, Nepal in 2015, i.e. similar location to install the field hospitals. This indicates that we have got "a stable solution", according to the three optimization models.

Keywords: Combinatorial optimization, Meta-heuristics, Algorithms, Operations research, Logistics..

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Chapter 1

Introduction

Robust Optimization (RO) is a methodology to deal with data uncertainty where the variability of the data is represented by deterministic values [Aissi et al., 2009; Kasperski et al., 2005; Kouvelis and Yu, 1997]. It emerged in the late sixties to deal with financial problems [Gupta and Rosenhead, 1968; Rosenhead et al., 1972] and has been applied, in general, as a way to self-protection against undesirable impacts due to vague approximations or incomplete, imprecise, or ambiguous data. Readers are referred to Roy [2010] for a survey on different uses of Robust Optimization in the field of Operations Research. Moreover, the surveys of Aissi et al. [2009]; Bertsimas et al. [2015]; Coco et al. [2014b]; Gabrel et al. [2014]; Kasperski and Zieliński [2016] cover Robust Optimization strategies and theoretical issues, and the studies of Conde [2012]; Chassein and Goerigk [2015]; Kasperski and Zieliński [2006]; Montemanni et al. [2004, 2007]; Siddiqui et al. [2011] focus on exact and heuristic algorithms to Robust Optimization problems.

Many robust counterparts of classical optimization problems have been studied in the literature, such as the Robust Shortest Path Problem [Coco et al., 2014a; Karasan et al., 2001], the Robust Minimum Spanning Tree Problem [Pérez-Galarce et al., 2014; Yaman et al., 2001], the Robust Assignment Problem [Pereira and Averbakh, 2011] and the Robust Shortest Path Tree Problem [Carvalho et al., 2016]. These problems are NP-Hard [Aissi et al., 2009], despite the fact that their deterministic counterparts are solved in polynomial time. RO problems whose deterministic counterparts are already NP-Hard have also been studied, such as the robust traveling salesman problem [Montemanni et al., 2007], the robust set covering problem [Pereira and Averbakh, 2013], the robust knapsack problem [Furini et al., 2015], the robust restricted shortest path problem [Assunção et al., 2017], and the robust vehicle routing problem [S.-Charris et al., 2015, 2016]. It is worth mentioning that several challenges regarding the design of algorithms and mathematical formulations are observed in RO problems whose classical counterpart is NP-Hard, since the complexity of computing the cost of a single solution is at least the computational complexity of the classical counterpart, and the mathematical formulations for such problems have an exponential number of constraints.

Uncertain data is modeled here as an interval of continuous values. In this approach, any realization of a single value for each parameter is considered as a scenario that can happen. The objective is to find a solution that is efficient for all scenarios, usually refereed to as a robust solution. The RO criterion used in this work to classify a solution as robust or not is the *min-max regret*. It was chosen because it is one of the most studied RO criterion [Aissi et al., 2009] and it is less-conservative than the *min-max* criterion [Soyster, 1973]. It was proposed by Wald [1939] for the game theory and was adapted to RO by Yu and Yang [1997]. In this thesis, RO approaches are developed and applied to *min-max regret* robust covering problems. Computational experiments are carried out on classical instances and on instances obtained from a real application found in disaster relief logistics.

This thesis investigates two problems: the *Min-max regret* Weighted Set Covering Problem (*min-max regret* WSCP) and the *min-max regret* Maximum Covering Location Problem (*min-max regret* MCLP). The former is a generalization of the Weighted Set Covering Problem (WSCP) [Edmonds, 1962], where the cost of each column j is modeled as an interval $[l_j, u_j]$, with $0 \leq l_j \leq u_j$, and the objective function consists in finding the subset of columns with the smallest maximum regret. The latter is a generalization of the Maximal Coverage Location Problem (MCLP) [Church and Velle, 1974], where the benefit of each column j is modeled as an interval $[l_j, u_j]$, and the objective function consists in finding the subset of columns, with a maximum cardinality T, that has the smallest maximum regret. The *min-max regret* MCLP was motivated by an application that emerged in a research project that is dedicated to optimize large scale operations after major disasters (e.g. earthquakes that hit Kathmandu, Nepal in 2015) [OLIC, 2015]. In this application, T field hospitals must be located after large-scale emergencies, subject to uncertainties associated with the number of inhabitants requiring medical care. Both the *min-max regret* WSCP and *min-max regret* MCLP), as well as WSCP and MCLP, are defined in the next chapter.

In this thesis, mathematical formulations, exact algorithms and heuristic methods are developed and applied to the *min-max regret* WSCP and to the *min-max regret*-MCLP. Concerning exact methods, three algorithms are developed: two Benders-Like Decomposition algorithms and a Branch-and-Cut (B&C) algorithm [Mitchell, 2002]. Concerning heuristic methods, two Scenario-based Algorithms [Coco et al., 2015; Kasperski and Zieliński, 2006], a Path Relinking [Glover and Laguna, 1993], a Pilot Method [Voss et al., 2005], and a linear programming based heuristic [Dantzig, 1963] are also proposed. All these methods are designed so that they can be generalized to other *min-max regret* optimization problems.

The remaining of this thesis is organized as follows:

- Chapter 2 presents the formal definition of WSCP and MCLP, as well as their robust counterparts, *i.e.* the *min-max regret* WSCP and the *min-max regret* MCLP.

- Chapter 3 gives a literature review on covering problems and robust optimization problems, as well as applications of these problems to disaster logistics.

- Chapter 4 is dedicated to integer linear programming formulations for the *min-max regret* MCLP, as well as other two alternative models for optimizing an application in post-disaster relief.

- In Chapter 5, exact and heuristic methods are proposed and applied to the *min-max regret* MCLP and the *min-max regret* MCLP.

- Computational experiments on the exact and heuristic algorithms proposed are reported and Analyzed in Chapter 6.

- The conclusions of this thesis, as well as opportunities for further research, are discussed in Chapter 7.

Chapter 2

Robust covering problems

This chapter introduces the notation used and defines the problems studied in this thesis. The deterministic counterparts of the *min-max regret* WSCP and the *min-max regret* MCLP are described in Section 2.1. Then, the two robust optimization problems addressed in this thesis are formally defined in Sections 2.2 and 2.3.

2.1 Deterministic Counterparts

Covering problems are one of the most studied combinatorial optimization problems [Caprara et al., 2000; Edmonds, 1962; Farahani et al., 2012]. The classical Weighted Set Covering Problem (WSCP) was introduced by Edmonds [1962] and is NP-hard [Garey and Johnson, 1979]. Let $\{a_{ij}\}$ be a matrix with a line-set N and a column-set M, where each column $j \in M$ is associated with a cost $c_j \geq 0$. WSCP consists in finding a subset $X \subseteq M$ whose the sum of the columns' cost is the minimum, and that every line in N is covered by at least one column in X. An example of a WSCP instance is found in Table 2.1, where the lines and the columns correspond respectively to the line-set and the column-set of $\{a_{ij}\}$. The optimal solution is given by columns $\{1, 3\}$, and the solution total cost is equal to 9.

The Maximal Covering Location Problem (MCLP) was introduced by Church and Velle [1974] and is NP-hard [Garey and Johnson, 1979]. Let $\{a_{ij}\}$ be a matrix with a line-set N and a column-set M, where each column $j \in M$ is associated with a benefit $b_j \geq 0$. Given a constant $T < |M|$, MCLP consists in finding a subset $X \subseteq M$, with $|X| \leq T$ whose the sum of the columns' benefit is the maximum, and every line in N is covered by at least one column in X. An example of a MCLP instance is found in Table 2.2, where the lines and the columns stand respectively for the line-set and

	1	2	3	4
1	1	1		
2	1			1
3	1			1
4		1	1	
5			1	1
6	1			1
7		1	1	
8			1	1
9		1	1	
c_j	3	8	6	4

Table 2.1: An example of an instance of WSCP with $M = 4$ and $N = 9$. The optimal solution $X^* = \{1, 3\}$ is highlighted.

the column-set of $\{a_{ij}\}$. The optimal solution for $T = 2$ is given by columns $\{2, 4\}$ and the total cost solution is equal to 12.

	1	2	3	4
1	1	1		
2	1			1
3	1			1
4		1	1	
5			1	1
6	1			1
7		1	1	
8			1	1
9		1	1	
b_j	3	8	6	4

Table 2.2: An example of an instance of MCLP with $M = 4$, $N = 9$ and $T = 2$. The optimal solution $X^* = \{2, 4\}$ is highlighted.

WSCP and MCLP have practical applications in scheduling [Caprara et al., 1999; Fisher and Rosenwein, 1989; Smith, 1988], metallurgy [Vasko et al., 1989], emergency medical services [Brotcorne et al., 2003; Gendrau et al., 1997; Li et al., 2011], post-disaster relief [Jia et al., 2007a,b; W.Yia and Özdamarb, 2007], facility location [Farahani et al., 2012; Schilling et al., 1993], reserve selection [Church et al., 1996; Snyder and Haight, 2016; Tong and Murray, 2009], geography [Murray, 2005] and etc. Some of these applications are very often subject to uncertain data. This motivates the study of such problems with uncertainties.

2.2 The min-max regret WSCP

The *min-max regret* WSCP, introduced by Pereira and Averbakh [2013], is a robust counterpart of WSCP where the cost of each column is uncertain and modeled as an interval of possible values. Let us consider N, M and $\{a_{ij}\}$ as previously defined, and $[l_j, u_j]$ be an interval with the minimum and the maximum expected cost for column $j \in M$. Moreover, a scenario $s \in S$ is an assignment of a single value $c_j^s \in [l_j, u_j]$ for each column $j \in M$, where S is the set of all possible combinations of values for the columns' cost. It is worth noticing that there are infinitely many scenarios in S. As WSCP, the *min-max regret* WSCP consists in finding $X \subseteq M$, such that every line in N is covered by at least one column in X. The main difference relies on the cost of each column which is uncertain. As a consequence, the objective function is adapted to address this issue.

The *min-max regret* objective function of WSCP is defined as follows. Let Γ be the set of feasible solutions, and $\omega^s(X) = \sum_{j \in X} c_j^s$ be the cost of a solution $X \in \Gamma$ for the scenario $s \in S$, where c_j^s is the cost of column $j \in M$ in s. The *regret* $\rho^s(X)$ of a solution $X \in \Gamma$ for a scenario $s \in S$ is defined as the difference between $\omega^s(X)$ and $\omega^s(Y^s)$, where Y^s is the optimal solution for the scenario s, i.e. the regret of using X instead of Y^s if scenario s occurs. The *min-max regret* WSCP aims at finding the solution X^*, given in Equation (2.1), that minimizes the maximum regret.

$$X^* = \arg\min_{X \in \Gamma} \max_{s \in S} \left\{ \omega^s(X) - \omega^s(Y^s) \right\} \tag{2.1}$$

An example of a solution of *min-max regret* MCLP is presented in Table 2.3. In Table 2.3 (a), the solution $X = \{1, 3\}$ is highlighted. A scenario $s \in S$ is displayed in Table 2.3(b). In this case, $Y^s = \{2, 4\}$ and the regret of X in s is $\omega^s(X) - \omega^s(Y^s) = 1$, where $\omega^s(X) = 7 + 4 = 11$ and $\omega^s(Y^s) = 7 + 3 = 10$. It is noteworthy that computing the regret of a *min-max regret* MCLP instance is NP-Hard, since evaluating a unique scenario corresponds to solve a WSCP.

Although there are an infinite number of scenarios in S, given a solution $X \in \Gamma$, the scenario $s(X)$, where the regret of X is the maximum can be computed in polynomial time for any min-max regret robust optimization problem, whose classical counterpart is a minimization problem [Karasan et al., 2001]. In these cases, $s(X)$ is the scenario where $c_j^{s(X)} = u_j$, for all $j \in X$, and $c_j^{s(X)} = l_j$, for all $j \in M \setminus X$, i.e. $s(X)$ is the scenario in which all columns in X have the largest possible cost and all other columns have the smallest possible cost.

An example of a solution for the *min-max regret* WSCP is presented in Table

	1	2	3	4
1	1	1		
2	1			1
3	1			1
4		1	1	
5			1	1
6	1			1
7		1	1	
8			1	1
9		1	1	
$[l_j, u_j]$	[5,8]	[3,7]	[3,4]	[6,9]

(a)

	1	2	3	4
1	1	1		
2	1			1
3	1			1
4		1	1	
5			1	1
6	1			1
7		1	1	
8			1	1
9		1	1	
c_j^s	7	3	4	7

(b)

Table 2.3: (a) An example of an instance of *min-max regret* WSCP with $M = 4$ and $N = 9$. The solution $X = \{1, 3\}$ is highlighted. (b) A scenario s of the instance shown in (a). The optimal solution of s, given by $X^* = \{2, 4\}$, is highlighted.

2.4(a), where the solution $X = \{1, 3\}$ is highlighted. The scenario $s(X)$ is displayed in Table 2.4(b). In this case, $Y^{s(X)} = \{2, 4\}$ and the regret of X in s is $\omega^{s(X)}(X) - \omega^{s(X)}(Y^{s(X)}) = 3$, where $\omega^{s(X)}(X) = 8 + 4 = 12$ and $\omega^{s(X)}(Y^{s(X)}) = 3 + 6 = 9$, and this is the optimal solution for this instance.

	1	2	3	4
1	1	1		
2	1			1
3	1			1
4		1	1	
5			1	1
6	1			1
7		1	1	
8			1	1
9		1	1	
$[l_j, u_j]$	[5,8]	[3,7]	[3,4]	[6,9]

(a)

	1	2	3	4
1	1	1		
2	1			1
3	1			1
4		1	1	
5			1	1
6	1			1
7		1	1	
8			1	1
9		1	1	
c_j^s	8	3	4	6

(b)

Table 2.4: (a) An example of an instance of *min-max regret* WSCP with $M = 4$ and $N = 9$. The solution $X = \{1, 3\}$ is highlighted. (b) Scenario $s(X)$ for the solution $X = \{1, 3\}$. The optimal solution of $s(X)$, given by $X^* = \{2, 4\}$, is highlighted.

2.3 The min-max regret MCLP

The *min-max regret* MCLP, proposed in this thesis, is a robust counterpart of MCLP, where the benefit of each column is uncertain and modeled as an interval of possible values. Let N, M, $\{a_{ij}\}$ and T be as defined before, and $[l_j, u_j]$ be an interval with the minimum and the maximum benefit expected for column $j \in M$. A scenario $s \in S$ is defined as an assignment of a single value $b_j^s \in [l_j, u_j]$ for each column $j \in M$, where S is the set of all possible values for the columns' benefit. It is worth noticing that there are infinitely many scenarios in S. As MCLP, *min-max regret* MCLP consists in finding $X \subseteq M$, such that $|X| \leq T$ and every line in N is covered by at least one column in X. The main difference relies on the benefit of each column which is uncertain. As a consequence, the objective function is adapted to address this issue.

Let Δ be the set of feasible solutions, and $\psi^s(X) = \sum_{j \in X} b_j^s$ be the benefit of a solution $X \in \Delta$ for the scenario $s \in S$, where b_j^s is the benefit of column $j \in M$ in s. The *regret* of a solution $X \in \Delta$ for a scenario $s \in S$ is defined as the difference between $\psi^s(Y^s)$ and $\psi^s(X)$, where Y^s is the optimal solution for the scenario s, i.e. the regret of using X instead of Y^s if scenario s happens. The *min-max regret* MCLP aims at finding the solution X^* that minimizes the maximum regret, as displayed in Equation (2.2).

$$X^* = \arg\min_{X \in \Delta} \max_{s \in S} \left\{ \psi^s(Y^s) - \psi^s(X) \right\} \tag{2.2}$$

Table 2.5(a) illustrates an example for the *min-max regret* MCLP, where the solution $X = \{1, 3\}$ is highlighted. A scenario $s \in S$ is displayed in Table 2.5(b). In this case, $Y^s = \{2, 4\}$ and the regret of X in s is $\psi^s(Y^s) - \psi^s(X) = 1$, where $\psi^s(Y^s) = 7 + 4 = 11$ and $\psi^s(X) = 7 + 3 = 10$. It is worth noticing that computing the regret of a solution on a single scenario is NP-Hard, since solving this problem in one scenario relies on solving MCLP, in order to compute y^s.

Although, there are an infinite number of scenarios in S, given a solution $X \in \Delta$, the scenario $s(X)$ where the regret of X is the maximum can be computed in polynomial time for any min-max regret robust optimization problem whose classical counterpart is a maximization problem [Furini et al., 2015]. In these cases, $s(X)$ is the scenario where $b_j^{s(X)} = l_j$, for all $j \in X$, and $b_j^{s(X)} = u_j$, for all $j \in M \setminus X$, i.e. $s(X)$ is the scenario in which all columns in X have the smallest possible benefit and all other columns have the largest possible benefit.

Table 2.6(a) presents an example used to compute the maximum regret of a solution for the *min-max regret* MCLP. Let us consider the solution $X = \{1, 3\}$ which

	1	2	3	4
1	1	1		
2	1			1
3	1			1
4		1	1	
5			1	1
6	1			1
7		1	1	
8			1	1
9		1	1	
$[l_j, u_j]$	[3,7]	[5,8]	[6,9]	[3,4]

(a)

	1	2	3	4
1	1	1		
2	1			1
3	1			1
4		1	1	
5			1	1
6	1			1
7		1	1	
8			1	1
9		1	1	
b_j^s	3	7	7	4

(b)

Table 2.5: (a) An example of an instance of *min-max regret* MCLP with $M = 4$, $N = 9$ and $T = 2$. The solution $X = \{1, 3\}$ is highlighted. (b) A scenario s of the instance shown in (a). The optimal solution of s, given by $X^* = \{2, 4\}$, is highlighted.

is highlighted in Table 2.6(a). The scenario $s(X)$ is displayed in Table 2.6(b). In this case, $Y^{s(X)} = \{2, 4\}$ and the maximum regret of X is $\psi^{s(X)}(Y^{s(X)}) - \psi^{s(X)}(X) = 3$, where $\psi^{s(X)}(Y^{s(X)}) = 8+4 = 12$ and $\psi^{s(X)}(X) = 3+6 = 9$, that is the optimal solution for this instance.

	1	2	3	4
1	1	1		
2	1			1
3	1			1
4		1	1	
5			1	1
6	1			1
7		1	1	
8			1	1
9		1	1	
$[l_j, u_j]$	[3,7]	[5,8]	[6,9]	[3,4]

(a)

	1	2	3	4
1	1	1		
2	1			1
3	1			1
4		1	1	
5			1	1
6	1			1
7		1	1	
8			1	1
9		1	1	
b_j^s	3	8	6	4

(b)

Table 2.6: (a) An example of an instance of *min-max regret* MCLP with $M = 4$, $N = 9$ and $T = 2$. The solution $X = \{1, 3\}$ is highlighted. (b) Scenario $s(X)$ for the solution $X = \{1, 3\}$. The optimal solution of $s(X)$, given by $X^* = \{2, 4\}$, is highlighted.

The *min-max regret* MCLP can model a real-life problem where field hospitals must be located after large-scale emergencies, such as the earthquakes that hit Kathmandu, Nepal on April 2015 [OLIC, 2015]. In this problem, T field hospitals must be settled in a set M of sites (columns) that cover a set N of neighborhoods (lines). The objective is to maximize the number of wounded inhabitants that have access to field hospitals. This is a typical application where robust optimization can be used, where scenarios (bounds) are defined according to the number of people in a neighborhood that may be affected.

Chapter 3

Literature review

This chapter provides a literature review on covering and robust optimization problems, as well as on applications of covering and facility location problems in disaster logistics. The related works on deterministic covering and facility location problems are reviewed in Section 3.1 while the related works on covering and facility location problems under uncertainty are revised in Section 3.2. Then, covering and facility location problems applied to in post-disaster logistics are reviewed in Section 3.3.

3.1 Related works on deterministic covering and location problems

WSCP [Edmonds, 1962], was one of the first covering problems introduced in the Operations Research literature and it is proved to be an NP-Hard [Garey and Johnson, 1979] problem. Edmonds [1962] focused on theoretical results for WSCP, proposing also a mathematical formulation.

Several methods were developed to WSCP. Chvátal [1979] introduced a greedy heuristic, without numerical experiments. Balas and Ho [1980] presented an algorithm that couples a set of primal and dual heuristics to a subgradient optimization method [Shor et al., 1985]. It proves optimality for instances with up to 200 lines and 2 000 columns. Beasley [1987] proposed a linear programming algorithm [Dantzig, 1963] based on a dual ascent [Bertsekas, 1999] and a subgradient optimization method which also encountered optimal solutions for instances containing a maximum of 200 lines and 2 000 columns. However, it performed better than the algorithm proposed by Balas and Ho [1980]. Beasley [1990a] introduced a heuristic based in Lagrangian relaxation [Wolsey, 1998] and subgradient optimization that solved instances with of 1 000 lines and 10 000 columns. Feo and Resende [1989] introduced a Greedy Randomized Adaptive Search procedure (GRASP) [Feo and Resende, 1995] meta-heuristic which found feasible solutions for instances up to 9 801 lines and 243 columns. Caprara et al. [1999] proposed a Lagrangian heuristic able to prove optimality for large instances with up to 5 000 lines and 1 million of columns. Furthermore, Lan et al. [2007] proposed a meta-heuristic, where the worst columns are penalized and the column priority policy is selected randomly. This heuristic was applied to instances containing 1 000 lines and 10 000 columns, and it performed better than the ones proposed by Beasley [1990a]; Caprara et al. [1999].

The MCLP was introduced by Church and Velle [1974] and was proved to be NP-hard by [Garey and Johnson, 1979]. Church and Velle [1974] presented a formulation and developed a linear programming algorithm and greedy heuristics to MCLP which solved instances of 55 lines and columns.

Exact and heuristic algorithms were developed to the MCLP. Downs and Camm [1996] introduced an hybrid algorithm by coupling dual-based solution methods and greedy heuristics to a B&B framework. This algorithm found optimal solutions for instances with up to 400 lines and 60 columns. Galvão and ReVelle [1996] developed a Lagrangian heuristic and a subgradient optimization algorithm which proved optimality to instances containing a maximum of 150 lines and columns. Resende [1998] introduced a GRASP which found high quality solutions for instances having 10 000 lines and 1 000 columns. Galvão et al. [2000] proposed a heuristic based in a surrogate approach [Dyer, 1980] that solved instances with up to 900 lines and columns and performed better than the Lagrangean heuristic introduced by Galvão and ReVelle [1996]. Xia et al. [2009] proposed a genetic algorithm, a tabu search and a simulated annealing and compared these methods with the ones developed by Downs and Camm [1996]; Resende [1998]. Computational experiments showed that the simulated annealing found, in average the best solutions among all heuristics. Recently, Máximo et al. [2017] extended the GRASP [Resende, 1998] by adding a learning stage [Kohonen et al., 2001]. This algorithm solved instances up to 3 038 lines and columns and found better solutions than previous works.

The Weighted Set Partitioning Problem (WSPP), the Maximum Coverage Problem (MCP) and the Weighted Vertex Covering problem (WVCP) are closely related to WSCP and MCLP. Thus, they are briefly described below in order to provide a wider perspective on a class of covering problems, that also were applied in the past to disaster logistics applications.

Given N, M, $\{a_{ij}\}$ and c_j as previously defined. The WSPP was introduced in Garfinkel and Nemhauser [1969] and consists in finding a subset $X \subseteq M$ with the minimum cost, such that every line in N is covered by exactly one column in X. Practical applications of WSPP are described in Darby-Dowman and Mitra [1985]; Ryan [1992]. This problem is NP-hard [Garey and Johnson, 1979], and exact and heuristic algorithms are proposed in Chu and Beasley [1998]; Garfinkel and Nemhauser [1969]; Yeh [1986].

The Maximum Coverage Problem (MCP) was defined in Nemhauser et al. [1978] as an extension of MCLP. Let us consider N, M, $\{a_{ij}\}$ and T as defined before. MCP consists in finding a subset $X \subseteq M$, with $|X| \leq T$, that maximizes the number of lines covered by X. Practical applications of MCP are described in Akhtar and Sahoo [2015]; Hammar et al. [2013]. This problem is NP-hard [Garey and Johnson, 1979], and exact and heuristic algorithms are proposed in Ageev and Sviridenko [1999]; Hochbaum [1997]; Nemhauser et al. [1978].

The Weighted Vertex Covering problem (WVCP) was proposed in Bar-Yehuda and Even [1981]. Let $G = (V, E)$ be a connected graph with a set V of nodes and a set E of edges, where each vertex $j \in V$ is associated with a cost c_j. WVCP consists in finding a subset $X \subseteq V$ with the minimum cost, such that each edge in E is covered by at least a vertex in X. Practical applications of WVCP are described in Bar-Yehuda et al. [2006]; Guha et al. [2002]. This problem is NP-hard [Bar-Yehuda and Even, 1981], and exact and heuristic algorithms are proposed in Bar-Yehuda and Even [1981]; Hochbaum [1982]; Gomes et al. [2006].

3.2 Related works on covering and locations problems under uncertainty

Contributions from the literature, focusing on covering problems with uncertain parameters are detailed below. Mainly, the uncertain parameters used for covering problems are associated with the columns' costs, or the probability to choose a column. Knapsack models considering uncertain data are also reviewed in this section since the MCLP uses a constraint to limit the number of selected columns, which is close to a knapsack constraint.

The pioneering work dealing with the *min-max regret* covering problems is the one of Pereira and Averbakh [2013] that introduced a linear formulation, exact methods based on Benders Decomposition and branch-and-cut (B&C), a genetic algorithm and a hybrid heuristic to the *min-max regret* WSCP. Computational experiments showed that B&C had the best performance among the exact methods and the hybrid heuristic produced the best upper bounds among the heuristic methods.

The Probabilistic Set Covering Problem (PSCP) is a generalization of the Set Covering Problem [Edmonds, 1962] that was proposed in Beraldi and Ruszczyński [2002]. Let N, M and $\{a_{ij}\}$ be as previously defined. Let also $\lambda \in (0, 1)$ be a constant and ξ be a $\{0, 1\}$-random vector. PSCP consists in finding a subset $X \subseteq M$ with minimum cost, such that every line in N is covered by at least ξ columns in X with a probability of at least λ. Practical applications of PSCP are described in Beraldi et al. [2004]; Nair and Miller-Hooks [2010]. This problem is NP-hard [Beraldi and Ruszczyński, 2002], and exact and heuristic algorithms are proposed in Beraldi and Ruszczyński [2002]; Beraldi et al. [2004].

Facility location problems have been treated in the context of disaster logistics. However, a few works handle facility location problems with uncertain parameters. In particular, the *min-max regret* Facility Location Problem (*min-max regret* FLP) is a generalization of the FLP [Hakimi, 1964, 1965] that was proposed in Snyder [2006]. Let B be a set of customers and F a set of facilities, where each facility $f \in F$ is associated with a cost interval $[l_f, u_f]$. A scenario $s \in S$ is an assignment of costs $c^s_f \in [l_f, u_f]$ for every facility $f \in F$. Let S be the set of possible scenarios, and $R^s(X)$ be the regret of using a solution X instead of the optimal weighted set covering solution Y^s in the scenario $s \in S$. The *min-max regret* FLP consists in finding a robust solution $X^* \subseteq F$ with the smallest maximum regret over all scenarios, such that all customers in D are serviced by a subset of open facilities. This problem is NP-hard [Snyder, 2006], and exact and heuristic algorithms were proposed in Alumur et al. [2012].

The *min-max regret* Knapsack Problem (*min-max regret* KP) is a generalization of the KP [Cormen et al., 2009]. It was introduced by Furini et al. [2015], which also proved that this problem is NP-hard. Let $W \in \mathbb{N}$ be the knapsack capacity and I be a set of items, where each item $i \in I$ is associated with a weight $w_i \in \mathbb{N}$ and a benefit interval $[l_i, u_i]$. A scenario $s \in S$ is an assignment of benefits $b^s_i \in [l_i, u_i]$ for every item $i \in I$. Let S and $R^s(X)$ be as defined above. The *min-max regret* KP consists in finding a robust solution $X^* \subseteq I$ with the smallest maximum regret over all scenarios, such that $\sum_{i \in X^*} w_i \leq W$. Exact and heuristic algorithms were proposed in Furini et al. [2015].

Another strategy, named Budget Uncertainty (BU) [Bertsimas and Sim, 2004], has also been applied to solve problems related to this thesis. In such an approach, uncertainties are also handled in the constraints and the methods try to maintain the solutions in a predefined and acceptable limit for the uncertain parameters (budget). Readers are referred to Bertsimas et al. [2011, 2015] for further information on this strategy. The KP has been extended and modeled by means of BU, given the BU-KP Bertsimas and Sim [2004], defined as follows. Let $W \in \mathbb{N}$ be the knapsack capacity and

I be a set of items, where each item $i \in I$ is associated with a weight interval $[l_i, u_i]$ and a benefit $b_i \in \mathbb{N}$. A scenario $s \in S$ is an assignment of weights $w_i^s \in [l_i, u_i]$ for every item $i \in I$. Given a scenario $t \in S$ and a set of scenarios $\varphi \in S$, where $s \in \varphi$ only if $w_i^s \leq w_i^t \ \forall i \in I$, the BU-KP aims at finding a subset $X^* \subseteq I$ with maximum benefit, such that W is not exceeded neither in t nor in any scenario $s \in \varphi$. This problem is NP-hard [Bertsimas and Sim, 2004] and algorithmic approaches were developed in Fischetti and Monaci [2012]; Lee et al. [2012]; Monaci et al. [2013].

The Budget Uncertainty Weighted Set Covering Problem (BU-WSCP) is a generalization of the WSCP [Edmonds, 1962] that was introduced in Lutter et al. [2017] and couples the robust optimization to the stochastic programming. Let $\{a_{ij}\}$, N, M, $[l_j, u_j]$ and S be as defined in Chapter 2 and λ be as described in PSCP. Given a scenario $t \in S$ with $k \leq |M|$ columns set in u_j and a set of scenarios $\beta \in S$ that contains all scenarios in which k or less columns are fixed in u_j. The BU-WSCP aims at finding a subset $X \subseteq M$ with the minimum cost sum in t, such that every line in N is covered by at least one column in X with a probability of at least λ and X is feasible for all scenarios in β. Lutter et al. [2017] proposed a linear formulation for the BU-WSCP.

3.3 Related works on applications of covering and location problems to disaster logistics

Several optimization models for local emergencies, such as fire, floods, local accidents, and large-scale emergencies rely on p-median [Altay and Green III, 2006; Caunhye et al., 2012; Diaz et al., 2013; Hakimi, 1965; Ortuño et al., 2013], p-center [Altay and Green III, 2006; Caunhye et al., 2012; Diaz et al., 2013; Hakimi, 1964; Ortuño et al., 2013], covering [Altay and Green III, 2006; Edmonds, 1962; Ortuño et al., 2013] and maximum covering [Church and Velle, 1974; Farahani et al., 2012; Leiras et al., 2014] problems. Those that are related to the *min-max regret* WSCP and the *min-max regret* MCLP are presented below.

The Location Set Covering Problem (LSCP), proposed by Toregas et al. [1971], applies an approach similar to WSCP for solving a post-disaster relief problem. LSCP is defined as follows. Let $\{a_{ij}\}$ be as previously defined, let D be a set of demand points and F be a set of facilities. LSCP consists in finding a subset $X \subseteq F$ with the minimum cost, such that every demand point in D is covered by at least one facility in F. It is worth mention that this problem is similar to WSCP, where $M \cong F$ and $N \cong D$. Toregas et al. [1971] also proposed a mathematical model and a linear programming algorithm that proves optimality to a real-world instance with 30 columns and lines.

Beraldi et al. [2004] couples the PSCP to a general facility location problem (GFLP), and addressed emergency medical services with uncertain demands. Let us consider N, M, $\{a_{ij}\}$ and λ as defined for the PSCP. Let also $E \subseteq M$ be a set of emergency medical facilities and H be a set of vehicles. PSCP-GFLP aims at minimizing the cost of placing all emergency medical facilities $e \in E$, such that at least $h \leq |H|$ vehicles are assigned to each facility and each line in N is covered with a probability of at least λ. Moreover, Beraldi et al. [2004] introduced a stochastic programming mathematical model used as a framework to solve similar problems. Computational experiments on instances proposed by Daskin [1983] showed the effectiveness and the reliability of the proposed model.

Dessouky et al. [2006] coupled a facility location and a vehicle routing problem to ensure the fast distribution of medical supplies in a post-disaster context. This problem is referred here as FL-VRP and its formulation minimizes the cost of the tours between a set of facilities and a set of demand points, such that all facilities should be placed at predetermined points. Computational experiments were performed in a real data instance from the Los Angeles County, in order to simulate an anthrax attack. The drawback is that numerical results were presented for a small instance with seven facilities and demand points.

Jia et al. [2007a,b] surveyed facility location and covering problems applied to small and large scale emergencies. Jia et al. [2007a] proposed a mathematical model based in a facility location problem (FLP-LSE) to deal with large scale emergencies. Computational experiments were performed using the instance of Dessouky et al. [2006]. Results to simulate a dirty bomb, anthrax and smallpox attacks were given. Jia et al. [2007b] introduced a mathematical model based in a maximum covering problem (MC-LSE) to handle large scale emergencies. Numerical results for the Dessouky et al. [2006] instances demonstrated the model efficiency by testing it for different levels of gravity for anthrax attacks.

Huang et al. [2010] proposed a variation of the p-center problem, named Large-Scale Emergency Center Problem (LSECP), to deal with large-scale emergencies. LSECP introduced the idea of redundancy by assigning more than one facility to each demand point, since some facilities may become inaccessible after a large-scale disaster. The authors also proposed a formulation and a dynamic programming algorithm. Computational experiments on using OR-Library [Beasley, 1990b] instances showed that the dynamic programming algorithm produced better solutions than the proposed mathematical model implemented in the IBM/ILOG CPLEX solver.

Horner and Downs [2010] proposed a variation of the Capacitated Facility Location Problem to deal with Hurricane Emergencies, refereed here as CFLP-HE. CLFP-HE formulation is given by the capacitated facility location problem mathematical model with additional constraints to take into account the protocols set by Florida's Comprehensive Emergency Plan. Computational experiments for a realistic instance based on the data of a small city in Florida, United States concluded that the distribution infrastructure used to provide relief and the assumptions regarding the population needs aid impact the accessibility to critical supplies (water, food, medicines, etc.).

Degel et al. [2015] introduced a variation of MCLP, referred here as MCLP-EMS, to deal with emergency medical services. The mathematical model of MCLP-EMS is given by MCLP's formulation with additional constraints that considers the relocation of emergency medical services, time windows and economic factors, such as the number of personnel and ambulances required during a day. Computational experiments made in a realistic data instance based in the city of Bochum, Germany, concluded that the proposed mathematical model achieves its objectives and leads to good solutions in terms of cost-effectiveness and quality of emergency care.

Recently, Duhamel et al. [2016] introduced a Multi-Period Facility Location Problem for Large Scale Emergencies, named here MPFLP-LSE, which improves the humanitarian aid distribution in a post-disaster context. The authors proposed a non-linear mathematical model to the MPFLP-LSE in which constraints of human, financial and material resources are added to the classical formulation of the multi-period facility location problem. To solve this problem, the authors proposed a decomposition algorithm where the master problem is addressed by a non-linear solver and the sub-problem is solved by a black-box heuristic and a Variable Neighborhood Descent local search [Hansen and Mladenović, 2001]. Computational experiments considered several post-disaster scenarios in case of floods in a real data instance from the city of Belo Horizonte, Brazil. Results indicated that increasing the number of distribution centers do not necessarily improve the number of people serviced. Others analysis were done to evaluate the method performance and the impact on the population receiving aid.

Table 3.1 summarizes the main characteristics that distinguish the robust optimization problems studied in this thesis from those presented in the literature. The first column identifies each approach. The next three columns indicate whenever a Deterministic Version (DV) is handled or else the reference addresses uncertainties, i.e. Uncertain Version (UV). In case of UV, two main approaches are distinguished, i.e. Robust Optimization (RO) and Stochastic Programming (SP). The following three columns inform about the constraints considered. The fifth column, called Coverage, is checked if the problems have coverage constraints, while sixth and seventh columns, referred respectively as location and knapsack ones. The last column shows whether realistic instances (RI) are used to assess the performance of the algorithms proposed to solve the problems. The characteristics of *min-max regret* WSCP and *min-max regret* MCLP are given in the last two lines. One can observe that *min-max regret* WSCP does not have practical applications. Also, to the best of our knowledge, *min-max regret* MCLP is the first problem in the literature to deal with both coverage and knapsack constraints under parameter uncertainty. We do not handle facility location constraints, because neighborhoods are not assigned to specific field hospitals, *i. e.*, the inhabitants are not constrained to the closest field hospital in their neighborhood.

Problem	DV	UV		Coverage	Location	Knapsack	RI
		RO	SP				
WSCP [Edmonds, 1962]	•			•			
MCLP [Church and Velle, 1974]	•			•		•	
WSPP [Garfinkel and Nemhauser, 1969]	•			•		•	
LSCP [Toregas et al., 1971]	•				•		
MCP [Nemhauser et al., 1978]	•					•	
PSCP [Beraldi and Ruszczyński, 2002]			•		•		
PSCP-FLP [Beraldi et al., 2004]			•		•		•
Budgeted Uncertainty KP [Bertsimas and Sim, 2004]		•				•	
Min-max regret FLP [Snyder, 2006]		•			•		
FLP-VP [Dessouky et al., 2006]	•				•		•
FLP-LSE [Jia et al., 2007a]	•			•	•		•
MC-LSE [Jia et al., 2007b]	•			•			•
CFL-HE [Horner and Downs, 2010]	•				•		•
LSECP [Huang et al., 2010]	•				•		•
Budgeted Uncertainty SCP [Lutter et al., 2017]		•	•	•			
Min-max regret KP [Furini et al., 2015]		•				•	
MCLP-EMS [Degel et al., 2015]	•			•	•	•	•
MPFLP-LSE [Duhamel et al., 2016]					•		•
Min-max regret WSCP [Pereira and Averbakh, 2013]		•		•			
Min-max regret MCLP		•		•		•	•

Table 3.1: Overview on the characteristics of the *min-max regret* WSCP, the *min-max regret* MCLP and related problems from the literature.

Chapter 4

Mathematical formulations

In this chapter, the mathematical formulations for the problems tackled by this thesis are presented. The WSCP [Edmonds, 1962] and the *min-max regret* WSCP [Pereira and Averbakh, 2013] mathematical formulations are reviewed in Section 4.1. Then, the mathematical formulations of the MCLP [Church and Velle, 1974], the *max-min upper scenario* MCLP, the *max-min lower scenario* MCLP and the *min-max regret* MCLP are described in Section 4.2.

4.1 WSCP and min-max regret WSCP formulations

Given N, M and $\{a_{ij}\}$ as defined in Section 2.2, we refer to c_j as the cost of selecting column $j \in M$. When c_j is uncertain, c_j^s we denote c_j^s as the cost of selecting the column $j \in M$ in the scenario s. Each solution $X \in \Gamma$ of the WSCP and of the *min-max regret* WSCP is associated with a characteristic vector of dimension $|M|$, such that X is represented by a vector x, with $x_j = 1$ if column $j \in X$ belongs to the solution, and $x_j = 0$ otherwise.

The WSCP formulation is given by the objective function (4.1) and constraints (4.2) and (4.3). Equation (4.1) aims at finding the solution $X \subseteq M$ with minimum cost when every line $i \in N$ is covered by at least one column $j \in x$. Inequalities (4.2) ensure that every line in N is covered by at least one column in M. Besides, the domain of variables x is defined in (4.3). It is worth mentioning that the set Γ of feasible solutions is given by constraints (4.2) and (4.3).

$$\min \quad \sum_{j \in M} c_j x_j \qquad s.t. \tag{4.1}$$

$$\sum_{j \in M} a_{ij} x_j \geq 1 \qquad \forall i \in N \tag{4.2}$$

$$x \in \{0,1\}^{|M|} \tag{4.3}$$

First, the formulation above is extended and a mixed integer linear programming (MILP) formulation is provided to the *min-max regret* WSCP. It is formulated by the objective function (4.4) and the constraints (4.2) and (4.3), where $\omega^s(y^s) = \sum_{j \in M} c_j^s y_j^s$ is the cost of the optimal solution y^s of the WSCP for the scenario s.

$$\min_{X \in \Gamma} \max_{s \in S} \quad \{\omega^s(x) - \omega^s(y^s)\} \tag{4.4}$$

Next, the objective function (4.4) is rewritten as (4.5), in order to explicitly compute the values of $\omega^s(x)$ and $\omega^s(y^s)$.

$$\min_{X \in \Gamma} \max_{s \in S} \quad \left\{ \sum_{j \in M} c_j^s x_j - \min_{y \in \Gamma} \left\{ \sum_{j \in M} c_j^s y_j \right\} \right\} \tag{4.5}$$

Then, let the scenario $s(x)$ given by Equation (4.6) be the scenario where the regret of x is maximum. As aforementioned, Karasan et al. [2001] proves that $s(x)$ can be obtained in polynomial time for any *min-max regret* RO problem (e.g. *min-max regret* WSCP), whose classical counterpart is a minimization problem (e.g. WSCP). For the *min-max regret* WSCP, $s(x)$ is the scenario where $c_j^{s(x)} = u_j$, when $x_j = 1$, and $c_j^{s(x)} = l_j$ otherwise. In the following, the objective function (4.5) is further rewritten in Equation (4.7), in such a way that only the worst case scenario $s(x)$ is considered. In this case, the term (a) of (4.7) gives the cost of x in $s(x)$ while the term (b) gives the cost of the optimal solution in scenario $s(x)$.

$$s(x) = \arg\max_{s \in S} \quad \left\{ \sum_{j \in M} c_j^s x_j - \min_{y \in \Gamma} \{ \sum_{j \in M} c_j^s y_j \} \right\} \tag{4.6}$$

$$\min_{X \in \Gamma} \quad \left\{ \underbrace{\sum_{j \in M} u_j x_j}_{(a)} - \underbrace{\min_{y \in \Gamma} \left\{ \sum_{j \in M} (l_j + (u_j - l_j) x_j) y_j \right\}}_{(b)} \right\} \tag{4.7}$$

Finally, equation (4.7) has been linearized by Pereira and Averbakh [2013] following the work of Montemanni et al. [2007]. Term (b) is replaced by a free variable

θ in (4.8) and the resulting Mixed Integer Linear Programming (MILP) formulation is given by the objective function (4.8), the constraints (4.9) and (4.10) that ensure $\theta = \omega^{s(x)}(y^{s(x)})$ as well as constraints (4.2) and (4.3). It is worth noting that the number of constraints (4.9) grows exponentially with the cardinality of M. Also, it is worth mentioning that there is no approach in the literature to derive a compact MILP formulation for interval min-max regret problems where the classical problem counterpart is NP-Hard.

$$\min_{X \in \Gamma} \quad \Big\{ \sum_{j \in M} u_j x_j - \theta \Big\} \qquad s.t. \tag{4.8}$$

$$\theta \leq \sum_{j \in M} l_j y_j + \sum_{j \in M} y_j \left(u_j - l_j\right) x_j \qquad \forall y \in \Gamma \tag{4.9}$$

$$\theta \quad free \tag{4.10}$$

WSCP is polynomially reducible to the *min-max regret* WSCP by making $l_j = u_j = c_j$ for every column $j \in |M|$. Consequently, the *min-max regret* WSCP is NP-Hard. This problem is harder than the WSCP since computing the cost of a single solution $X \in \theta$ requires solving a WSCP instance in the scenario $s(X)$. Therefore, the decision version of the *min-max regret* WSCP is in P if and only if $P = NP$.

4.2 MCLP and min-max regret MCLP formulations

Given N, M, $\{a_{ij}\}$ and $T < |M|$ as defined in Section 2.3, we refer to b_j as the benefit of selecting column $j \in M$. When b_j is uncertain, we denote b_j^s as the benefit of selecting the column $j \in M$ in the scenario s. Each solution $X \in \Delta$ of the MCLP and the *min-max regret* MCLP is associated with a characteristic vector of dimension $|M|$, such that X is represented by a vector x, with $x_j = 1$ if $j \in X$, and $x_j = 0$ otherwise.

The MCLP formulation is given by the objective function (4.11) and the constraints (4.12) to (4.14). The objective function (4.11) aims at finding the solution $X \in \Delta$ with the maximum benefit. Inequalities (4.12) ensure that every line in N is covered by at least one column in M. Moreover, constraint (4.13) enforces that at most T columns are selected. Besides, the domain of variables x is defined in (4.14). It is worth mentioning that the set Δ of feasible solutions is formulated by constraints (4.12) to (4.14).

$$\max \quad \sum_{j \in M} b_j x_j \qquad s.t. \tag{4.11}$$

$$\sum_{j \in M} a_{ij} x_j \geq 1 \qquad \forall i \in N \tag{4.12}$$

$$\sum_{j \in M} x_j \leq T \tag{4.13}$$

$$x \in \{0, 1\}^{|M|} \tag{4.14}$$

In addition to the *min-max regret* MCLP, two alternative robust optimization problems are formulated to deal with the uncertainty on the value of b_j. They are based on the *max-min* criterion [Soyster, 1973], which is very frequently used in the literature. In the first problem, all columns $j \in M$ are set to the lowest value of the interval and the resulting scenario is called lower scenario. In the second, all columns $j \in M$ are set to the highest value of the interval and the resulting scenario is called upper scenario. The idea of these formulations is to provide alternative robust optimization models which can be applied to major disasters.

The *max-min upper scenario* MCLP maximizes the number of people having access to a field hospital in the scenario where the number wounded people is the maximum. The rationale behind this problem is that the optimization of the worst case scenario results in a solution that is efficient even in the scenario where the field hospitals are the most overcrowded.

The *max-min lower* MCLP maximizes the number of people that have access to a field hospital in the scenario where the number wounded people is the minimum. The rationale behind this problem is that the optimial solution of the *max-min lower scenario* MCLP ensures that at least a lower bound lb of wounded inhabitants have access to field hospitals in all possible scenarios.

4.2.1 Max-min upper scenario MCLP

The *max-min upper scenario* MCLP is formulated as follows. The objective function (4.15) computes the number of wounded inhabitants that have access to field hospitals in the scenario where the number of wounded inhabitants is the maximum. Therefore, the *max-min upper scenario* MCLP ILP is given by (4.15) and the constraints (4.12) to (4.14).

$$\max \max_{s \in S} \sum_{j \in M} b_j^s x_j \tag{4.15}$$

The scenario where the number of wounded inhabitants is the maximum is the one with $b_j^s = u_j$. Therefore, the objective function (4.15) can be rewritten as (4.16).

$$\max \sum_{j \in M} u_j x_j \tag{4.16}$$

It can be seen that any instance of the *max-min upper* is equivalent to an instance of the MCLP with $b_j = u_j$. Consequently, the *max-min upper* is NP-Hard and any algorithm to the MCLP can be used to solve the *max-min upper*.

4.2.2 Max-min lower scenario MCLP

The *max-min lower* MCLP is formulated as follows. The objective function (4.17) computes the number of wounded inhabitants looking for field hospitals in the scenario where the number of wounded inhabitants is the minimum. Therefore, the *max-min lower* MCLP ILP is given by the objective function (4.17) and the constraints (4.12) to (4.14),

$$\max \min_{s \in S} \sum_{j \in M} b_j^s x_j \tag{4.17}$$

It holds that the scenario where the number of wounded inhabitants is the minimum is the one with $b_j^s = l_j$.

$$\max \sum_{j \in M} l_j x_j \tag{4.18}$$

It can also be seen that any instance of the *max-min lower scenario* MCLP is equivalent to an instance of the MCLP with $b_j = l_j$. Consequently, the *max-min lower scenario* MCLP is also NP-Hard and any algorithm to the MCLP can be used to solve the *max-min lower scenario* MCLP.

4.2.3 Min-max regret MCLP

In this section, the formulation of the MCLP is extended and a mixed integer linear programming (MILP) formulation is provided to the *min-max regret* MCLP. It is formulated by the objective function (4.19) and the constraints (4.12) to (4.14), where $\psi^s(y^s) = \sum_{j \in M} b_j^s y_j^s$ is the benefit of the optimal solution y^s of the MCLP for the scenario s.

$$\min_{x \in \Delta} \max_{s \in S} \quad \{\psi^s(y^s) - \psi^s(x)\} \tag{4.19}$$

First, the objective function (4.19) is rewritten as (4.20), in order to explicitly compute the values of $\psi^s(y^s)$ and $\psi^s(x)$.

$$\min_{x \in \Delta} \max_{s \in S} \quad \left\{ \max_{y \in \Delta} \left\{ \sum_{j \in M} b_j^s y_j \right\} - \sum_{j \in M} b_j^s x_j \right\} \tag{4.20}$$

Then, let $s(x)$, given in Equation (4.21), be the scenario where the regret of x is the maximum. The authors of Furini et al. [2015] showed how to compute $s(x)$ in polynomial time for any min-max regret robust optimization problem, whose classical counterpart is a maximization problem. In the case of the *min-max regret* MCLP, $s(x)$ is the scenario where $b_j^{s(x)} = l_j$, when $x_j = 1$, and $b_j^{s(x)} = u_j$ otherwise. According to this result, the objective function (4.20) can be rewritten as (4.22), in such a way that only the worst case scenario $s(x)$ is considered. In this case, the term (a) of (4.22) gives the benefit of the optimal solution in scenario $s(x)$, while the term (b) gives the benefit of x in $s(x)$.

$$s(x) = \arg\max_{s \in S} \quad \left\{ \max_{y \in \Delta} \{ \sum_{j \in M} b_j^s y_j \} - \sum_{j \in M} b_j^s x_j \right\} \tag{4.21}$$

$$\min_{x \in \Delta} \quad \left\{ \underbrace{\max_{y \in \Delta} \left\{ \sum_{j \in M} (u_j + (l_j - u_j) x_j) y_j \right\}}_{(c)} - \underbrace{\sum_{j \in M} l_j x_j}_{(d)} \right\} \tag{4.22}$$

Finally, equation (4.22) is linearized following the approach proposed by Furini et al. [2015]. Term (c) is replaced by a free variable μ and the MILP formulation for the *min-max regret* MCLP is given by the objective function (4.23), constraints (4.24) and (4.25) which ensure that $\mu = \psi^{s(x)}(y^{s(x)})$, and constraints (4.12) to (4.14). It is worth noting that the number of constraints (4.24) grows exponentially with the cardinality of M.

$$\min_{x \in \Delta} \quad \left\{ \mu - \sum_{j \in M} l_j x_j \right\} \qquad s.t. \tag{4.23}$$

$$\mu \geq \sum_{j \in M} u_j y_j + \sum_{j \in M} y_j \left(l_j - u_j \right) x_j \qquad \forall y \in \Delta \tag{4.24}$$

$$\mu \quad free \tag{4.25}$$

MCLP is polynomially reducible to the *min-max regret* MCLP by making $l_j = u_j = b_j$ for every column $j \in |M|$. Consequently, the *min-max regret* MCLP is also NP-hard. Besides, this problem is harder than the MCLP since computing the cost of a single solution $X \in \Delta$ requires solving a MCLP instance in the scenario $s(X)$. Therefore, the decision version of the *min-max regret* MCLP is in P if and only if $P = NP$.

Chapter 5

Algorithms

This chapter describes the exact and heuristic algorithms proposed in this thesis for both *min-max regret* WSCP and *min-max regret* MCLP. The exact algorithms proposed by [Pereira and Averbakh, 2013] for the *min-max regret* WSCP have been reproduced in this thesis and they are detailed in Section 5.1. In the following, they are generalized for the *min-max regret* MCLP. In the sequel, heuristic algorithms for the *min-max regret* WSCP and the *min-max regret* MCLP are proposed and described in Section 5.2.

5.1 Exact algorithms

The first exact algorithm proposed for the *min-max regret* WSCP [Pereira and Averbakh, 2013] relies on a cutting plane algorithm inspired by the Benders Decomposition [Benders, 1962]. It is usually referred in the literature as the Benders-like Decomposition algorithm (BLD). It is similar to the methods applied to solve the min-max regret Traveling Salesman Problem [Montemanni et al., 2007], the min-max regret Knapsack Problem [Furini et al., 2015], and the min-max regret Restricted Shortest Path Problem [Assunção et al., 2016].

5.1.1 Benders-like decomposition algorithm for the min-max regret WSCP

The BLD for the *min-max regret* WSCP is based on the mathematical model (4.2)-(4.3) and (4.8)-(4.10). As explained in Chapter 4, the number of constraints (4.9) increases exponentially with the number of columns. Thus, they are relaxed and replaced by (5.1) in the master problem as follows. Let $\Gamma^h \subseteq \Gamma$ be the set of solutions that induce the constraints (5.1). The algorithm works as follows. At each iteration, a new constraint is separated from $\Gamma \setminus \Gamma^h$, by solving a WSCP sub-problem, and added to the master problem. BLD stops when the lower bound obtained by solving the master problem is equal to the upper bound or when a time limit is reached.

$$\theta \leq \sum_{j \in M} u_j y_j + \sum_{j \in M} y_j \left(l_j - u_j\right) x_j \quad \forall y \in \Gamma^h \tag{5.1}$$

The pseudo-code of BLD is shown in Algorithm 1. Let $\rho(X) = \omega^{s(X)}(X) - \omega^{s(X)}(Y^{s(X)})$ be the maximum regret of a solution $X \in \Gamma$, where $Y^{s(X)}$ is the optimal solution of WSCP in the scenario $s(X)$. In order to avoid an unbounded master problem, Γ^1 is initialized in lines 1 to 3 with two solutions, X^m and X^u, obtained by the Algorithm Mean Upper (AMU) [Kasperski and Zieliński, 2006] described in Section 5.2, as suggested by Montemanni et al. [2007]. The loop in lines 4 to 10 is performed until an optimal solution is found or when a time limit is reached. The master problem is run from Γ^h in line 5. Let (X^h, θ^h) be the optimal solution of this problem. We point out that X^h is feasible for the *min-max regret* WSCP, but the lower bound z obtained from the master problem may not be equal to $\rho(X^h)$, because the value of θ^h may not be equal to $\omega^{s(X)}(Y^{s(X)})$. Therefore, a WSCP sub-problem is run in line 6 in order to obtain the optimal solution $Y^{s(X^h)}$ for the scenario $s(X^h)$. This solution is added to Γ^{h+1} in line 7, which induces a new constraint (5.1) that cuts the solution (X^h, θ^h). The best known solution X^* is updated in line 8, and the iteration counter h is incremented in line 9. If $z = \rho(X^*)$, the optimal solution X^* is returned in line 11. The proof that this algorithm converges to an optimal solution is found in Assunção et al. [2017].

Input: $M, N, A, [l_j, u_j]\ \forall j \in M$
Output: X^*
1 $h \leftarrow 1$
2 $\{X^m, X^u\} \leftarrow \text{AMU}(M, N, A, [l_j, u_j]\ \forall j \in M)$
3 $\Gamma^h \leftarrow \{X^m, X^u\}$
4 **do**
5 $\quad (X^h, \theta^h, z) \leftarrow \text{MasterProblem}(M, N, A, [l_j, u_j]\ \forall j \in M, \Gamma^h)$
6 $\quad Y^{s(X^h)} \leftarrow \text{WSCP}(M, N, A, s(X^h))$
7 $\quad \Gamma^{h+1} \leftarrow \Gamma^h \cup \{Y^{s(X^h)}\}$
8 $\quad X^* \leftarrow \arg\min_{X \in \{X^h, X^*\}} \rho(X)$
9 $\quad h \leftarrow h + 1$
10 **while** $z < \rho(X^*)$ *and the time limit is not reached*;
11 **return** X^*

Algorithm 1: Pseudo-code of BLD for the *min-max regret* WSCP.

5.1.2 Benders-like decomposition for the robust MCLP

The BLD algorithm to the *min-max regret* MCLP is based on the formulation (4.12)-(4.14) and (4.23)-(4.25). As explained in Chapter 4, the number of constraints (4.24) increases exponentially with the number of columns. Thus, they are relaxed and replaced by (4.24) in the master problem as follows. Let $\Delta^h \subseteq \Delta$ be the set of solutions that induce the constraints (5.2). The algorithm works as follows. At each iteration, a new constraint is separated from $\Delta \setminus \Delta^h$, by solving a MCLP sub-problem, and added to the master problem. BLD stops when the lower bound obtained by solving the master problem is equal to the upper bound or when a time limit is reached.

$$\mu \geq \sum_{j \in M} u_j y_j + \sum_{j \in M} y_j \left(l_j - u_j\right) x_j \quad \forall y \in \Delta^h \tag{5.2}$$

The pseudo-code of BLD for the *min-max regret* MCLP is shown in Algorithm 2. Let $\rho(X) = \psi^{s(X)}(Y^{s(X)}) - \psi^{s(X)}(X)$ be the maximum regret of a solution $X \in \Delta$, where $Y^{s(X)}$ is the optimal solution of MCLP in the scenario $s(X)$. As in BLD for the *min-max regret* WSCP, Δ^1 is initialized with the solutions X^m and X^u, also returned by the AMU heuristic [Kasperski and Zieliński, 2006], in order to avoid an unbounded master problem. The loop in lines 4 to 10 is performed until an optimal solution is found or when a time limit is reached. The master problem is run from Δ^h in line 5. Let (X^h, μ^h) be the optimal solution of this problem. We point out that X^h is feasible for the *min-max regret* MCLP, but the lower bound z obtained from the master problem may not be equal to $\rho(X^h)$, because the value of μ^h may not be equal to $\psi^{s(X)}(Y^{s(X)})$. Therefore, a MCLP sub-problem is run in line 6 in order to obtain the optimal solution $Y^{s(X^h)}$ for the scenario $s(X^h)$. This solution is added to Δ^{h+1} in line 7, which induces a new constraint (5.2) that cuts the solution (X^h, μ^h). The best known solution X^* is updated in line 8, and the iteration counter h is incremented in line 9. If $z = \rho(X^*)$ in line 10, the optimal solution X^* is returned in line 11.

5.1.3 Extended Benders

Pereira and Averbakh [2013] demonstrated that the convergence of BLD may be slow, since only one cut is produced after the run of a difficult master problem. In order to speed up the convergence of BLD, Pereira and Averbakh [2013] developed an extension of BLD called Extended Benders (EB). EB follows a method introduced by Fischetti et al. [2010] where all incumbent solutions found by CPLEX are used to generate new

Input: $M, N, A, T, [l_j, u_j]\ \forall j \in M$
Output: X^*
$h \leftarrow 1$
$\{X^m, X^u\} \leftarrow \text{AMU}(M, N, A, T, [l_j, u_j]\ \forall j \in M)$
$\Delta^h \leftarrow \{X^m, X^u\}$
do
    $(X^h, \mu^h, z) \leftarrow \text{MasterProblem}(M, N, A, T, [l_j, u_j]\ \forall j \in M, \Delta^h)$
    $Y^{s(X^h)} \leftarrow \text{MCLP}(M, N, A, T, s(X^h))$
    $\Delta^{h+1} \leftarrow \Delta^h \cup \{Y^{s(X^h)}\}$
    $X^* \leftarrow \arg\min_{X \in \{X^h, X^*\}} \rho(X)$
    $h \leftarrow h + 1$
while $z < \rho(X^*)$ *and the time limit is not reached*;
return X^*

Algorithm 2: Pseudo-code of BLD for the *min-max regret* MCLP.

cuts (not only the optimal one), while the master problem is solved. A subproblem is solved for each incumbent solution and the solutions returned by each subproblem are added to the Master Problem. Therefore, the expected number of iterations in EB is smaller than of BLD.

The pseudo-code of EB proposed in this thesis for the *min-max regret* MCLP is shown in Algorithm 3. Let $\rho(X)$, (X^h, μ^h), X^m and X^u be as defined in Section 5.1.2. In order to avoid an unbounded master problem, Δ^1 is initialized as BLD. The loop in lines 4 to 12 is performed until an optimal solution is found or when a time limit is reached. The master problem is run from Δ^h in line 5. Let α^h be the set of incumbent solutions (X_i^h, μ_i^h) found in iteration h of the master problem. In EB, a MCLP subproblem is run in line 7 for each $(X_i^h, \mu_i^h) \in \alpha_h$ in order to obtain the optimal solution $Y^{s(X_i^h)}$ for the scenario $s(X_i^h)$. This solution is added to Δ^{h+1} in line 8, which induces a new constraint (5.2) that cuts each solution (X_i^h, μ_i^h). The best known solution X^* is updated in line 9, and the iteration counter h is incremented in line 11. If $z = \rho(X^*)$ in line 12, the optimal solution X^* is returned in line 13. EB can be straightforwardly extended to the *min-max regret* WSCP.

The Scenario Based Algorithm (SBA), described in Section 5.2.1, returns up to a hundred solutions, that can be inserted in Γ^h or Δ^h before the first iteration of EB. Thus, in Algorithm 3, SBA takes place of AMU in line 2. Therefore, the resulting algorithm, named SBA+EB starts with a larger Γ^1 or Δ^1 than EB.

The heuristic Path Relinking (PR), described in Section 5.2.2 can be called at the end of each iteration h to compare the solutions added to Γ^h or to Δ^h at h with all known solutions. Therefore, new solutions are added to Γ^h or to Δ^h at each iteration.

Input: $M, N, A, T, [l_j, u_j]\ \forall j \in M$
Output: X^*
1 $h \leftarrow 1$
2 $\{X^m, X^u\} \leftarrow \text{AMU}(M, N, A, T, [l_j, u_j]\ \forall j \in M)$
3 $\Delta^h \leftarrow \{X^m, X^u\}$
4 **do**
5     $(\alpha^h, \mu^h, z) \leftarrow \text{MasterProblem}(M, N, A, T, [l_j, u_j]\ \forall j \in M, \Delta^h)$
6   **foreach** $X_i^h \in \alpha^h$ **do**
7         $Y^{s(X_i^h)} \leftarrow \text{MCLP}(M, N, A, T, s(X_i^h))$
8         $\Delta^{h+1} \leftarrow \Delta^h \cup \{Y^{s(X_i^h)}\}$
9         $X^* \leftarrow \arg\min_{X \in \{X_i^h, X^*\}} \rho(X)$
10   **end**
11     $h \leftarrow h + 1$
12 **while** $z < \rho(X^*)$ *and the time limit is not reached*;
13 **return** X^*

Algorithm 3: Pseudo-code of EB for the *min-max regret* MCLP.

In algorithm 3, PR is added before 11 and the resulting algorithm is named PR+EB.

5.1.4 Branch and Cut

Montemanni et al. [2007] noticed that BLD may be computationally inefficient, because at each iteration of this algorithm an ILOG CPLEX branch-and-bound algorithm is run from scratch in order to solve the MILP formulation of the master problem. Thus, Montemanni et al. [2007] proposed an approach to the *min-max regret* Traveling Salesman Problem where only one instance of a branch-and-cut (B&C) algorithm is performed. B&C is an optimization method where the optimal solution is seek by means of a branch-and-bound tree in which cutting planes are applied to tighten the linear programming relaxations of the tree [Wolsey, 1998].

B&C was extended to the *min-max regret* WSCP in Pereira and Averbakh [2013] and to the *min-max regret* MCLP in this thesis. B&C for the *min-max regret* WSCP is based on the linear relaxation of the mathematical model (4.2)-(4.3), (4.8), (4.10) and (5.1) while the formulation (4.12)-(4.14), (4.23), (4.25) and (5.2) is the base to B&C for the *min-max regret* MCLP. As both B&C are similar, only the B&C for *min-max regret* MCLP is explained below. This algorithm starts with the formulation given by the subset $\Delta' = \Delta^1$ of constraints (5.2). When an integer solution (X', μ') is found in a node of the enumeration tree, a new solution $Y^{s(X')}$ is computed by solving the MCLP sub-problem in the scenario $s(X')$. Then, $Y^{s(X')}$ is added to Δ' and a new global cut is propagated to all active nodes in the branch-and-bound tree. Therefore, one does not need to restart the branch-and-bound algorithm from $\Delta' \cup \{Y^{s(X')}\}$. This algorithm is correct because for each solution X' found, a new constraint (4.24) is generated to enforce the correct value of μ' [Montemanni et al., 2007].

The pseudo-code of B&C for *min-max regret* MCLP is shown in Algorithm 4. Let $\rho(X)$, (X^h, μ^h), X^m and X^u be as defined in Section 5.1.2. In order to avoid an unbounded master problem, Δ^1 is initialized as BLD. The B&C framework in line 4 is performed until an optimal solution (X^*, μ^*) is found and optimal solution X^* is returned in line 5. B&C can be straightforwardly extended to the *min-max regret* WSCP.

```
Input: M, N, A, T, [l_j, u_j] ∀j ∈ M
Output: X*
1 {X^m, X^u} ← AMU(M, N, A, T, [l_j, u_j] ∀j ∈ M)
2 Δ^h ← {X^m, X^u}
3 (X*, μ*) ← Branch-and-Cut(M, N, A, T, [l_j, u_j])
4 return X*
```

Algorithm 4: Pseudo-code of B&C for *min-max regret* MCLP.

SBA also replaces AMU in B&C, *i.e.*, in line 1 of Algorithm 4, SBA substitutes AMU . In this case, the solutions found by SBA are given to Γ^h or Δ^h before running the B&C framework. Therefore, the resulting algorithm, named SBA+B&C, starts with an larger number of constraints (5.1) or (5.2) than B&C.

In B&C, PR can be called after an incumbent solution is found. Then, PR compares the last found solution with all known ones and, as consequence, more solutions are added to the Branch-and-bound tree. In Algorithm 4, PR is inserted into the Branch-and-Cut framework in line 3 and the resulting algorithm is called PR+B&C.

5.2 Heuristics

In this section, five heuristics are proposed to the *min-max regret* WSCP and the *min-max regret* MCLP: two scenario-based algorithms [Kasperski and Zieliński, 2006; Coco et al., 2015], a path relinking [Glover and Laguna, 1993], a pilot method [Voss et al., 2005] and a linear programming based heuristic [Dantzig, 1963] are detailed respectively in Sections 5.2.1, 5.2.2, 5.2.3 and 5.2.4. Except the one based in linear programming, they are guaranteed to be 2-approximative.

5.2.1 Scenario-based heuristics

Scenario-based heuristics for *min-max regret* combinatorial optimization problems consist in sampling a subset of scenarios and optimally solving the deterministic counterpart problem on each of these scenarios. Then, the maximum regret of each obtained solution is computed and the one with the smallest maximum regret is returned.

The *Algorithm Mean* (AM) heuristic for the *min-max regret* WSCP and the *min-max regret* MCLP are instantiations of the framework proposed in Kasperski and Zieliński [2006]. For the *min-max regret* MCLP, let the *mean scenario* s^m be the scenario where the cost of each column $j \in M$ is $c_j^m = (l_j + u_j)/2$. while for the *min-max regret* MCLP, s^m is the scenario where the benefit of each column $j \in M$ is $b_j^m = (l_j + u_j)/2$. AM consists in solving the classical combinatorial optimization problem (eg. WSCP or MCLP) in the scenario s^m and returning the obtained solution. The proof that this algorithm is 2-approximative for any *min-max regret* combinatorial optimization problem is below. Two variations of this approach are also proposed in Kasperski and Zieliński [2006]. The *Algorithm Upper* (AU) heuristic consists in solving the classical combinatorial optimization problem in the scenario s^u, where the cost of each column $j \in M$ is $b_j^u = u_j$, while the *Algorithm Mean Upper* (AMU) simply returns the best solution obtained by AM and AU.

Theorem 5.1 *AMU is 2-approximative for both the* min-max regret *WSCP [Kasperski and Zieliński, 2006] and the* min-max regret *MCLP.*

Preposition 5.1 *Let X^a and X^b be two feasible solutions for a problem studied in this thesis. Then, the following inequality holds [Kasperski and Zieliński, 2006]:*

$$\rho(X^a) \geq \sum_{j \in X^a \setminus X^b} u_j - \sum_{j \in X^b \setminus X^a} l_j \tag{5.3}$$

Proof. From the definition of maximum regret:

$$\rho(X^a) = \omega^{s(X^a)}(X^a) - \omega^{s(X^a)}(Y^{s(X^a)}) \tag{5.4}$$

$$\geq \omega^{s(X^a)}(X^a) - \omega^{s(X^a)}(X^b) \tag{5.5}$$

It is easy to see that

$$\omega^{s(X^a)}(X^a) - \omega^{s(X^a)}(X^b) = \sum_{j \in X^a \setminus X^b} u_j - \sum_{j \in X^b \setminus X^a} l_j$$

which together with (5.5), imply in (5.3) ■

Preposition 5.2 *Let again X^a and X^b be two feasible solutions. Then, the following inequality holds [Kasperski and Zieliński, 2006]:*

$$\rho(X^b) \leq \rho(X^a) + \sum_{j \in X^b \setminus X^a} u_j - \sum_{j \in X^a \setminus X^b} l_j \tag{5.6}$$

Proof.

It is easy to see that:

$$\omega^{s(X^b)}(X^b) = \omega^{s(X^a)}(X^a) + \sum_{j \in X^b \setminus X^a} u_j - \sum_{j \in X^a \setminus X^b} u_j \tag{5.7}$$

Then:

$$\omega^{s(X^b)}(Y^{s(X^b)}) \geq \omega^{s(X^a)}(Y^{s(X^a)}) - \sum_{j \in X^a \setminus X^b} (u_j - l_j) \tag{5.8}$$

Suppose that inequality (5.8) is false. Thus, let X^c be another feasible solution where $\omega^{s(X^b)}(X^c) = \omega^{s(X^b)}(Y^{s(X^b)})$. Thus,

$$\begin{aligned} \omega^{s(X^a)}(Y^{s(X^a)}) &> \omega^{s(X^b)}(X^c) + \sum_{j \in X^a \setminus X^b} (u_j - l_j) \\ &\geq \omega^{(s(X^a) \cup s(X^b))}(X^c) \\ &\geq \omega^{s(X^a)}(X^c) \end{aligned} \tag{5.9}$$

However, inequalities (5.9) contradict the definition of $\omega^{s(X^a)}(Y^{s(X^a)})$. Thus, subtracting (5.8) from (5.7) yields in (5.6). ■

Preposition 5.3 *Let X^{amu} be the solution returned by AMU and $\rho(X^{amu})$ be the regret of X^{amu}. Then, for any solution X^{amu}, it holds that $\rho(X^{amu}) \leq 2 \times \rho(X^*)$ [Kasperski and Zieliński, 2006].*

Proof. Since X^{amu} is the solution returned by AMU, it fulfills the inequality:

$$\frac{1}{2}\sum_{j \in X^{amu}} (l_j + u_j) \leq \frac{1}{2}\sum_{j \in X^*} (l_j + u_j) \tag{5.10}$$

Hence,

$$\sum_{j \in X^{amu} \setminus X^*} (l_j + u_j) \leq \sum_{j \in X^* \setminus X^{amu}} (l_j + u_j) \tag{5.11}$$

Which implies in,

$$\sum_{j \in X^* \setminus X^{amu}} u_j - \sum_{j \in X^{amu} \setminus X^*} l_j \geq \sum_{j \in X^{amu} \setminus X^*} u_j - \sum_{j \in X^* \setminus X^{amu}} l_j \tag{5.12}$$

Applying inequality (5.6) in equation (5.12):

$$\rho(X^{amu}) \leq \rho(X^*) + \sum_{j \in X^{amu} \setminus X^*} u_j - \sum_{j \in X^* \setminus X^{amu}} l_j \tag{5.13}$$

Inequality (5.3) together with (5.12) yield:

$$\begin{aligned} \rho(X^*) &\geq \sum_{j \in X^* \setminus X^{amu}} u_j - \sum_{j \in X^{amu} \setminus X^*} l_j \\ &\geq \sum_{j \in X^{amu} \setminus X^*} u_j - \sum_{j \in X^* \setminus X^{amu}} l_j \end{aligned} \tag{5.14}$$

Finally, equations (5.13) and (5.14) imply that $\rho(X^{amu}) \leq 2 \times \rho(X^*)$. ■

The pseudo-code of AMU is displayed in Algorithm 5. It works for both *min-max regret* WSCP and *min-max regret* MCLP. The optimal solutions X^m and X^u for the scenarios mean and upper are obtained in lines 1 and 2, respectively, the best known solution X' is updated in line 3 and returned in line 4. The worst case complexity of SBA is the same of solving an instance of MCLP.

Input: $q, M, N, A, T, [l_j, u_j]\ \forall j \in M$
Output: X'
1 $X^m \leftarrow$ MeanScenario(M, N, A, T)
2 $X^u \leftarrow$ UpperScenario(M, N, A, T)
3 $X' \leftarrow \arg\min_{X \in \{X^m, X^u\}} \rho(X)$
4 **return** X'

Algorithm 5: Pseudo-code of the AMU heuristic.

The Scenario Based Algorithm (SBA) heuristic is an instantiation of the framework proposed in Coco et al. [2015] and successfully applied in Carvalho et al. [2016]; Coco et al. [2016]. SBA is a generalization of AMU, where a set Q of scenarios, instead of a single one, is investigated. The algorithm consists of solving one instance of the problem (eg. WSCP or MCLP) for each scenario in Q, and returning the solution with the minimum maximal regret. Let s_p be the scenario where $b_j^{s_p} = \{l_j + (p \times (u_j - l_j))$ for each $j \in M\}$. We have that $Q = \{s_p \mid p = \frac{i}{q}$ and $i = 0, 1, 2, 3, \cdots, q\}$, where the number of SBA iterations q was set to 100. It is easy to see that, for an even value of q, the mean scenario is always investigated. Therefore, the solutions obtained by SBA are at least as good as those of AMU.

The pseudo-code of SBA for the *min-max regret* MCLP is displayed in Algorithm 6. At each of the q iterations of the for-loop in lines 1 to 6, a MCLP instance is solved in a specific scenario. The value of p and the scenario s_p are computed in lines 2 and 3, respectively. The optimal MCLP solution X^p for the scenario s_p is obtained in line 4, and the best known solution X' is updated in line 5. The best solution found by SBA is returned in line 6. One can observe that the SBA applied to the *min-max regret* WSCP is similar to the one of *min-max regret* MCLP.

Input: $q, M, N, A, T, [l_j, u_j]\ \forall j \in M$
Output: X'
1 **for** i ***from 0 to*** q **do**
2     $p \leftarrow i/q$
3   Let s_p be the scenario where $b_j^{s_p} \leftarrow l_j + p \times (u_j - l_j)$, $\forall j \in M$
4     $X^p \leftarrow \text{MCLP}(M, N, A, T, s_p)$
5     $X' \leftarrow \arg\min_{X \in \{X^p, X'\}} \rho(X)$
6 **end**
7 **return** X'

Algorithm 6: Pseudo-code of the SBA heuristic.

5.2.2 Path relinking

Path Relinking (PR) [Glover and Laguna, 1993] is a search heuristic that has been successfully applied to a number of optimization problems [Glover et al., 2000; Prins et al., 2006; Resende et al., 2010]. Given two solutions X^i and X^f, PR's main idea is to gradually transform X^i into X^f, by applying a set of different moves. This mechanism is motivated by the fact that different near-optimal solutions usually share good components found in local optima. For the *min-max regret* WSCP and the *min-max regret* MCLP, this is translated by two solutions that share a common subset of columns, *i.e.* in the *min-max regret* WSCP and the *min-max regret* MCLP, two solutions X^i and X^f may share a common subset of columns L where $L = X^i \cap X^f$ and $L \neq \emptyset$. Thus, PR uses this information to create a sequence of intermediate solutions between X^i and X^f, in the hope that better solutions will be find.

In the PR for the *min-max regret* WSCP and the *min-max regret* MCLP, the path of solutions between X^i and X^f is created using two different moves: (i) a column in X^i but not in X^f is removed from X^i and columns in X^f but not in X^i are added to X^i until all lines in X^i are covered and (ii) a column in X^f but not in X^i is introduced to X^i and the redundant columns in X^i are removed.

PR uses of a pool of feasible solutions that is initialized with all (up to a hundred) distinct solutions found by SBA. Moreover, new best solutions found during PR are also inserted in the pool. Two strategies of using this pool is investigated. In the first strategy (a), a path relinking is run from the best solution found by AMU for all the others in the pool. In the second strategy (b), a path relinking is run from every solution in the pool to all the others. The two different moves and the two strategies to use the pool are combined to generate four path relinking variations which are named PR-R-Best (i-a), PR-I-Best (i-b), PR-R-Any (ii-a) and PR-I-Any (ii-b). *PR* stands for Path Relinking, *R* and *I* mean, respectively, Remove a column first or Insert a column first, and *Best* and *All* indicate, respectively, if X^i is given only by the best solution found by SBA or all solutions.

The pseudo-code of PR-R-All for the *min-max regret* MCLP is displayed in Algorithm 6. Let P and X^{sba} be, respectively, the pool of solutions and the best solution found by SBA. X' is initialized with X^{sba} in line 1. The loop in lines 2-16 is performed for each solution in the pool. X^i is initialized in line 3 with the i-th solution in the pool. The loop in lines 4-15 is performed for each solution in the pool, except X^i and X^f. X^f and the column-set W containing all columns $j \in X^i$ and $j \notin X^f$ are initialized in lines 5 and 6, respectively. The loop in lines 7-14 is performed for each column in W. First, a column $l \in W$ is removed from X^i, in line 9. Then, columns $l \in X^f$ and $l \notin X^i$ are added to X^i in line 8. Finally, the best known solution X' is updated in line 10 and, if X' is not in P, then it is added to the pool in line 12. The best solution found by PR-R-All is returned in line 17.

The pseudo-code of PR-I-All for the *min-max regret* MCLP is displayed in Algorithm 8. Let P and X^{sba} be defined as above. X' is initialized with X^{sba} in line 1. The loop in lines 2-16 is performed for each solution in the pool. X^i is initialized in line 3. The loop in lines 4-15 is performed for each solution in the pool, except X^i. X^f and the column-set Z containing all columns $j \in X^f$ and $j \notin X^i$ are initialized in lines 5 and 6, respectively. The loop in lines 7-14 is performed for each column in Z. First, a column $l \in Z$ is added to X^i, in line 8. Then, columns $l \in X^i$ and $l \notin X^f$ are

```
   Input: P, X^sba, M, N, A, T, [l_j, u_j] ∀j ∈ M
   Output: X'
1  X' ← X^sba
2  for i from 0 to |P| do
3  |  X^i ← P_i
4  |  for k from i to |P| do
5  |  |  X^f ← P_k
6  |  |  W ← Compare(X^i, X^f)
7  |  |  for l from 0 to |W| do
8  |  |  |  Remove(X^i, W[l])
9  |  |  |  X^i ← AddColumns(l ∈ X^f and l ∉ X^i)
10 |  |  |  X' ← arg min_{X∈{X^p,X'}} ρ(X)
11 |  |  |  if X' ∉ P then
12 |  |  |  |  P ← X'
13 |  |  |  end
14 |  |  end
15 |  end
16 end
17 return X';
```

Algorithm 7: PR-R-All pseudo-code.

removed from X^i in line 9. Finally, the best known solution X' is updated in line 10 and, if X' is not in P, then it is added to the pool in line 12. The best solution found by PR-R-All is returned in line 17.

In PR-R-Best and PR-I-Best, the loop 1-12 is run only once with $X^i \leftarrow X^{SBA}$ in line 3. Moreover, all Path Relinking strategies can be straight forwardly extended to the *min-max regret* WSCP.

5.2.3 Pilot Method

Pilot Method (PM) [Duin and Voss, 1999; Voss et al., 2005] is a metaheuristic that uses a greedy constructive guiding heuristic H to build a new and more efficient heuristic H' and works as follows. Given a constructive heuristic, it will iteratively insert one element at a time in a partial solution. However, instead of using a local greedy criterion to evaluate the cost of inserting an element in the solution, the criterion used by H' consists in (i) inserting the element individually in the solution (ii) performing the heuristic H until a feasible solution is found, and (iii) using the cost of this solution as the greedy cost of inserting the element. At each iteration, these three steps are performed for all candidate elements and the one with the best greedy cost is inserted on the solution.

```
Input: P, X^{sba}, M, N, A, T, [l_j, u_j] ∀j ∈ M
Output: X'
X' ← X^{sba}
for i from 0 to |P| do
    X^i ← P_i
    for k from i to |P| do
        X^f ← P_k
        Z ← Compare(X^i, X^f)
        for l from 0 to |Z| do
            Add(X^i, Z[l])
            X^i ← RemoveColumns(l ∈ X^i and l ∉ X^f)
            X' ← arg min_{X∈{X^p,X'}} ρ(X)
            if X' ∉ P then
                P ← X'
            end
        end
    end
end
return X';
```

Algorithm 8: PR-I-All pseudo-code.

A survey on PM heuristics is found in Voss et al. [2005]. The PM was successfully used for solving NP-Hard combinatorial optimization problems, such as the traveling salesman problem [Duin and Voss, 1999], the Steiner tree problem [Duin and Voss, 1994, 1999], spanning tree problems [Martins, 2007; Xiong et al., 2006], container loading problems [Eley, 2002], network designing problems [Höller et al., 2008], and scheduling problems [Bertsekas and Castanon, 1999; Fink and Voss, 2003; Meloni et al., 2004]. Coco et al. [2014a] developed a PM framework for *minmax regret* problems and applied it to solve the *minmax regret* Shortest Path Problem. In this thesis, the PM proposed by Coco et al. [2014a] is extended to solve the *min-max regret* WSCP and the *min-max regret* MCLP. As suggested by Coco et al. [2014a], the heuristic AM [Kasperski and Zieliński, 2006] works as guiding heuristic in both algorithms.

The pseudo-code of PM for the *min-max regret* MCLP is presented in Algorithm 9. The algorithm inputs are: $M, N, A, T, [l_j, u_j]\ \forall j \in M$, defined in Chapter 2. The partial (or guiding) solution X' and the best known feasible solution X^{PM} are initialized at line 1. The loop on lines 2-12 is performed while X' is not feasible, *i.e.*, while all lines are not covered and less than T columns are chosen. Let $i \in N$ be a line and let $\delta^+(i)$ be the set of columns that cover line i. The uncovered line i with the highest $\delta^+(i)$ is identified at line 3. The loop on lines 4-9 is performed for each column $j \in \delta^+(i)$. The MCLP formulation using the mean scenario $s^m \in S$ is run at line 5 and returns a feasible solution X'_v which contains all columns in X'. Next, $\rho(X'_j)$ is used as the greedy cost of inserting column v in the solution X'. Then, if $\rho(X'_j)$ is smaller than the current iteration's best greedy cost $\rho(X'_{j^*})$ or else if the latter is not set yet (line 7), the column $j^* \in \delta^+(i)$ with the smallest maximum regret and its respective covering X'_{j^*} are updated in line 8. Afterwards, j^* is inserted in the end of X' at line 10. PM returns the best solution found throughout the heuristic X^{PM}, which is not necessarily X'. Therefore, the former is updated at line 11, and returned at line 13. PM can be straight forwardly extended to the *min-max regret* WSCP.

Input: $M, N, A, T, [l_j, u_j]\ \forall j \in M$
Output: X^{PM}
1 $X' \leftarrow \emptyset$ and $X^{PM} \leftarrow \emptyset$
2 **while** X' *is not a feasible solution* **do**
3   Let $i \in N$ be the line with the highest number of columns to cover it.
4   **for** $j \in \delta^+(i)$ **do**
5         $X'_j \leftarrow$ MCLPFormulation (j, X', s^m)
6   **if** $\rho(X'_j) < \rho(X'_{j^*})$ *or* $X'_{j^*} = \emptyset$ **then**
7             $j^* \leftarrow v$ and $X'_{j^*} \leftarrow X'_v$
8   **end**
9   **end**
10   Insert j^* at the end of X'
11     $X^{PM} \leftarrow \arg\min_{X \in \{X'_{j^*}, X^{PM}\}} \rho(X)$
12 **end**
13 **return** X^{PM};

Algorithm 9: Pseudo-code of PM for the *min-max regret* MCLP.

5.2.4 Linear Programming Heuristic

In this thesis, the Linear Programming Heuristic (LPH) proposed in Assunção et al. [2017] for the *min-max regret* Restricted Shortest Path problem is extended. As mentioned before, there is no modeling approach in the literature that provides compact formulations for *min-max regret* optimization problems, whose deterministic counterpart is NP-Hard. LPH consists in running an alternative compact formulation, which is obtained by replacing the problem's objective function by another one that does not give the exact maximum regret of a solution. However, the alternative objective function is guaranteed to return a value that is an upper bound to the solution's maximum regret. This heuristic is adapted to the *min-max regret* WSCP and to the *min-max regret* MCLP in sections 5.2.4.1 and 5.2.4.2, respectively. Again, each solution $X \in \Gamma$ of the *min-max regret* WSCP and of the *min-max regret* MCLP is associated with a characteristic vector of dimension $|M|$, such that X is represented by a vector x, with $x_j = 1$ if column $j \in X$ belongs to the solution, and $x_j = 0$, otherwise.

5.2.4.1 LPH for the min-max regret WSCP

First, the non-linear formulation of the *min-max regret WSCP* (4.2), (4.3), and (4.7) is rewritten below as (5.15)-(5.17). The ILP formulation of the subproblem that computes the optimal solution in the scenario $s(X)$ is highlighted in the term (f). It can be seen that this subproblem is equivalent to a WSCP, where the cost of a column $j \in M$ is equal to $l_j + (u_j - l_j)x_j$.

$$\min \quad \Big\{ \underbrace{\sum_{j \in M} u_j x_j}_{(e)} - \underbrace{\min_{y \in \Gamma} \Big\{ \sum_{j \in M} (l_j + (u_j - l_j)x_j) y_j \Big\}}_{(f)} \Big\} \qquad s.t. \qquad (5.15)$$

$$\sum_{j \in M} a_{ij} x_j \geq 1 \qquad \forall i \in N \qquad (5.16)$$

$$x \in \{0,1\}^{|M|} \qquad (5.17)$$

Then, the linear relaxation of the subproblem (f) is expanded in (5.18)-(5.20).

$$\min \quad \sum_{j \in M} (l_j + (u_j - l_j)x_j) y_j \qquad s.t. \qquad (5.18)$$

$$\sum_{j \in M} a_{ij} y_j \geq 1 \qquad \forall i \in N \qquad (5.19)$$

$$y_j \in [0,1] \qquad \forall j \in M \qquad (5.20)$$

Based on Dantzig's duality theory [Dantzig, 1963], Assunção et al. [2017] proved that an upper bound to the non-linear formulation of any *min-max regret* problem can be obtained by replacing the subproblem (f) by the dual of its linear relaxation. In the case of the *min-max regret* WSCP, the dual of formulation (5.18)-(5.20) is shown in (5.21)-(5.23).

$$\max \quad \sum_{i \in N} \nu_i \qquad s.t. \tag{5.21}$$

$$\sum_{i \in N} a_{ij}\nu_i \leq l_j + (u_j - l_j)x_j \qquad \forall j \in M \tag{5.22}$$

$$\nu_i \geq 0 \qquad \forall i \in N \tag{5.23}$$

Finally, the MILP formulation for the *min-max regret* WSCP is obtained by replacing the subproblem (f) in (5.15) by the dual relaxation (5.21)-(5.23). The resulting formulation is given by (5.16), (5.17), (5.22), (5.23) and (5.24). This formulation is compact, as the number of constraints (5.22) grows polynomially with the cardinality of M. The LPH heuristic for the *min-max regret* WSCP consists in solving and returning the best solution of this formulation.

$$\min \quad \sum_{j \in M} u_j x_j - \sum_{i \in N} \nu_i \tag{5.24}$$

5.2.4.2 LPH for the min-max regret MCLP

First, the non-linear formulation of the *min-max regret MCLP* (4.12)-(4.14), and (4.22) is rewritten below as (5.25)-(5.28). The MILP formulation of the subproblem that computes the optimal solution in the scenario $s(X)$ is highlighted in the term (g). It can be seen that this subproblem is equivalent to a MCLP, where the cost of a column $j \in M$ is equal to $u_j + (l_j - u_j)x_j$.

$$\min \quad \Bigg\{ \underbrace{\max_{y \in \Delta} \quad \Big\{ \sum_{j \in M} (u_j + (l_j - u_j)x_j)y_j \Big\}}_{(g)} - \underbrace{\sum_{j \in M} l_j x_j}_{(h)} \Bigg\} \qquad s.t. \tag{5.25}$$

$$\sum_{j \in M} a_{ij}x_j \geq 1 \qquad \forall i \in N \tag{5.26}$$

$$\sum_{j \in M} x_j \leq T \tag{5.27}$$

$$x \in \{0,1\}^{|M|} \tag{5.28}$$

Then, the linear relaxation of the subproblem (g) is expanded in (5.29)-(5.32).

$$\min \quad \sum_{j \in M} (l_j + (u_j - l_j)x_j)y_j \qquad s.t. \tag{5.29}$$

$$\sum_{j \in M} a_{ij} y_j \geq 1 \qquad \forall i \in N \tag{5.30}$$

$$\sum_{j \in M} y_j \leq T \tag{5.31}$$

$$y_j \in [0, 1] \qquad \forall j \in M \tag{5.32}$$

Based on Dantzig duality theory Dantzig [1963], Assunção et al. [2017] proved that an upper bound to the non-linear formulation of any *min-max regret* problem can be obtained by replacing the subproblem (g) by the dual of its linear relaxation. In the case of the *min-max regret* MCLP, the dual of formulation (5.29)-(5.32) is shown in (5.33)-(5.36).

$$\min \quad T\xi - \sum_{i \in N} \nu_i \qquad s.t. \tag{5.33}$$

$$\xi - \sum_{i \in N} a_{ij} \nu_i \geq u_j + (l_j - u_j)x_j \qquad \forall j \in M \tag{5.34}$$

$$\nu_i \geq 0 \qquad \forall i \in N \tag{5.35}$$

$$\xi \geq 0 \tag{5.36}$$

Finally, the MILP formulation for the *min-max regret* MCLP is obtained by replacing the subproblem (c) in (5.25) by the dual relaxation (5.33)-(5.36). The resulting formulation is given by (5.26)-(5.28), (5.34)-(5.36) and (5.37). This formulation is compact, as the number of constraints (5.34) grows polynomially with the cardinality of M. The LPH heuristic for the *min-max regret* MCLP consists in solving and returning the best solution for this formulation.

$$\min \quad (T\xi - \sum_{i \in N} \nu_i) - \sum_{j \in M} l_j x_j \tag{5.37}$$

Chapter 6

Computational experiments

Computational experiments were carried out on an Intel Core i7-4790K with 4.00 GHz clock and 16 GB of RAM, running Ubuntu Linux operating system version 14.04 LTS. Algorithms BLD, EB, B&C, AMU, SBA, PR, LPH, PM, SBA+EB, SBA+B&C, PR+EB and PR+B&C were implemented in C++ and compiled with GNU g++ version 4.8.2. The master and sub-problems of BLD, EB and B&C, as well as the WSCP and MCLP instances that arise from the heuristics, were solved using IBM/ILOG CPLEX[1] version 12.6.2 with default parameter settings.

Two test sets are used in the experiments. The first one was generated as suggested by Pereira and Averbakh [2013] and the instances of this set are generalizations of classical theoretical instances for the set covering problem. The cost interval of each column was generated according to Kasperski and Zieliński [2004]. As in Pereira and Averbakh [2013], the instance sets BKZ-4, BKZ-5, and BKZ-6 of Beasley [1990b] where used. Set BKZ-4 has 10 instances with $|N| = 1000$, $|M| = 200$, and the density of matrix $\{a_{ij}\}$ is 2%, while BKZ-5 set has 10 instances with $|N| = 2000$, $|M| = 200$, and the density of matrix $\{a_{ij}\}$ is also 2%. Besides, BKZ-6 set has 5 instances with $|N| = 1000$, $|M| = 200$, and the density of matrix $\{a_{ij}\}$ is 5%. As suggested by Kasperski and Zieliński [2004], for each column $j \in M$, the values of l_j and u_j are chosen, respectively, within the intervals $U[0, \lambda]$ and $U[l_j, l_j + \lambda]$, where $U[a, b]$ denotes a random number uniformly chosen in the range $[a, b]$, and the interval length λ is set to 1000. Moreover, in the *min-max regret* MCLP, the value of T was varied in $T = 0.1 \times |M|$, $T = 0.2 \times |M|$ and $T = 0.3 \times |M|$. Three instances were generated for each value of T and for each instance in sets BKZ-4, BKZ-5 and BKZ-6. Therefore, a total of 75 instances were generated in the experiments of the *min-max regret* WSCP and another 225 instances were proposed for the *min-max regret* MCLP.

[1] http://www-03.ibm.com/software/products/en/ibmilogcpleoptistud

The second set of instances is called *Kathmandu*. It is based on real data from the earthquakes that hit Kathmandu, Nepal in 2015. The data was obtained by the OLIC project OLIC [2015] from its partners in the International Charter on Space and Major Disasters (ICSMD). All instances have $|N| = 5$ hospitals and $|M| = 13$ potential sites to place the field hospitals, which were previously defined by ICSMD staff. The potential sites were chosen taken into account secured location, with water access, right dimensions, among others aspects. The matrix $\{a_{ij}\}$ describes whether a hospital $i \in N$ can be supported by a field hospital at site $j \in M$. The instances in this set differ from each other by the values of T and by the benefit intervals $[l_j, u_j]$. Let P_j^k be the total number of inhabitants in the k closest neighborhoods to site $j \in M$, we have that $l_j = (1-\beta) \times (\delta \times P_j^k)$ and $u_j = (1+\beta) \times (\delta \times P_j^k)$, where δ is a target percentage of the residents that would seek help at a field hospital placed at site j, and β is the degree of uncertainty, i.e. the relative width of the interval. We point out that in this case, the inhabitants of a specific neighborhood may be accounted for more than one site. We generated 60 instances named as Kat-k-δ-β as following. First, we sampled one interval from each of the combination of values for $k \in \{11, 21\}$, $\delta \in \{10\%, 20\%, 30\%, 40\%, 50\%\}$, and $\beta \in \{10\%, 30\%\}$. Then, from each of the 20 intervals, 3 instances varying the value of T from 3 to 5 field hospitals were generated.

The goals of the computational experiments are to analyze the performance of the algorithms proposed in this thesis for the *min-max regret* WSCP and the *min-max regret* MCLP on realistic and theoretical instances. In this chapter, each table presents only the main results found on each experiment. only a summary containing the main results of each experiment is presented. However, the full version of each table can be found in Appendix B. The results for the *min-max regret* WSCP and the *min-max regret* MCLP are displayed in 6.1 and 6.2, respectively.

6.1 Results for the min-max regret WSCP

The first experiment evaluates the exact algorithms BLD, EB and B&C [Pereira and Averbakh, 2013] on the set of 75 theoretical instances. The running times were limited to 900 seconds. The results are reported in Table 6.1. The first and the second columns present, respectively, the name and the size ($|G|$) of each instance set. The third column reports the number of optimal solutions ($|O|$) found by BLD in each instance set while the average relative optimality GAP $(\frac{\rho(X^{\text{BLD}})-z}{\rho(X^{\text{BLD}})})(\%)$ of each set in BLD is reported in the fourth column, where z is the lower bound obtained by solving at optimality the master problem in the last iteration of BLD. The fifth column shows

BLD's average running time, while the sixth column displays the average number of iterations, which, in BLD, is equal to the number of cuts added to the master problem. The same data is reported for EB and B&C, respectively, in columns 7 to 10 and 11 to 14. It is worth mentioning that, in B&C, the lower bound z is obtained by the linear relaxation of formulation (4.2)-(4.3), (4.8), (4.10) and (5.1) with Γ^h containing one cut for each integer solution found by B&C. In each line, the algorithm that performed better and the one that found the best average gaps are highlighted. As was shown by Pereira and Averbakh [2013], B&C is the algorithm which performed better among the exact algorithms, because it found the optimal solution in eight instances while EB encountered the optimal solution in four instances and BD did not return any optimal solution. Moreover, it is worth mentioning that the relative optimality gap of EB (10.63%) is, on average, smaller than that of B&C (11.02%). This behavior probably occurs because of the excessive number of constraints (5.1) unnecessarily generated during the run of B&C and, in consequence, the CPLEX takes more time to solve each node of the branch-and-bound tree.

A Time to Target plot (TTT-Plot) [Aiex et al., 2005] comparing the performance of BLD, EB and B&C [Pereira and Averbakh, 2013] for an instance set named BKZ-6s is displayed in Figure 6.1. Set BKZ-6s has 100 instances with $M = 500$, $N = 100$ and the intervals of each column $j \in M$ is similar to sets BKZ-4, BKZ-5 and BKZ-6. Set BKZ-6s has smaller instances than sets BKZ-4, BKZ-5 and BKZ-6, because, to guarantee a fair comparison between BLD, EB and B&C, it is important that all instances are solved to optimality in less than 900 seconds by all algorithms. Results indicate that the empirical probability of B&C be the fastest algorithm is 98% for the set BKZ-6s. Moreover, it can be seen that B&C found the optimal solution of approximately 90% of these instances in less than 10 seconds.

The second experiment compares the four proposed Path Relinking strategies for the set of 75 theoretical instances. The results are reported in Table 6.2. The first column presents the instance set's name. The second column reports the average percentage deviation $\frac{\rho(X^{\text{B\&C}})-\rho(X^{\text{PRRBEST}})}{\rho(X^{\text{B\&C}})}(\%)$ of the solutions provided by PR-R-Best relative to those of B&C while the third column shows the average running time for each set in PR-R-Best. The same data is reported for PR-R-Any, PR-I-Best and PR-I-Any, respectively, in columns 4 and 5, 6 and 7 and 8 and 9. Best average solutions found for each instance set are highlighted and a negative percentage deviation means that the heuristic found a better average feasible solutions than a 900 second run of the exact algorithm. Results indicate that PR-R-Any returned better solutions on average while PR-R-Best and PR-I-Best took less time than the other two strategies, as expected, since the number of solutions evaluated during their run is also smaller. It can also be

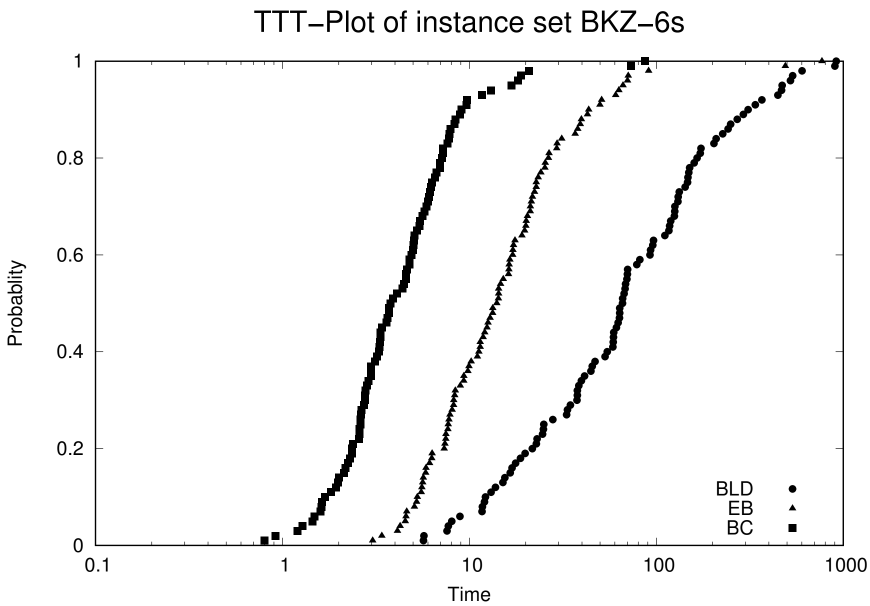

Figure 6.1: A TTT-Plot which compares the performance of BLD, EB and B&C to the *min-max regret* WSCP.

Instance	$\lvert G\rvert$	BLD $\lvert O\rvert$	BLD Gap	BLD T (s)	BLD Cuts	EB $\lvert O\rvert$	EB Gap	EB T (s)	EB Cuts	B&C $\lvert O\rvert$	B&C Gap	B&C T (s)	B&C Cuts
BKZ 4 a	10	0	20.76	900.00	23.40	0	14.12	900.00	144.50	0	13.74	900.00	253.30
BKZ 4 b	10	0	24.42	900.00	15.20	0	18.91	900.00	95.20	0	20.15	900.00	236.90
BKZ 4 c	10	0	21.71	900.00	18.90	0	15.08	900.00	115.60	0	16.34	900.00	810.70
BKZ 5 a	10	0	13.21	900.00	19.10	0	7.50	900.00	125.50	0	8.44	900.00	873.10
BKZ 5 b	10	0	14.43	900.00	15.00	0	9.07	900.00	94.40	0	10.09	900.00	810.70
BKZ 5 c	10	0	15.31	900.00	18.10	0	9.29	900.00	127.50	1	9.42	828.37	429.30
BKZ 6 a	5	0	8.42	900.00	17.00	3	2.21	759.58	100.60	4	0.76	484.61	316.80
BKZ 6 b	5	0	11.96	900.00	14.00	1	5.54	899.86	88.80	2	5.04	753.66	356.60
BKZ 6 c	5	0	10.94	900.00	15.20	0	3.76	900.00	115.80	1	3.17	806.78	140.00
Average	75	0	15.68	900.00	17.32	4	9.50	884.38	111.99	8	9.68	819.27	469.71

Table 6.1: Comparison among the exact algorithms proposed by Pereira and Averbakh [2013] to the set BKZ.

Instance	PR R Best Dev (%)	PR R Best T (s)	PR R Any Dev (%)	PR R Any T(s)	PR I Best Dev (%)	PR I Best T (s)	PR I Any Dev (%)	PR I Any T (s)
BKZ 4 a	0.96	22.73	0.86	85.97	1.06	23.46	1.03	84.20
BKZ 4 b	0.12	28.46	0.12	64.34	0.11	28.03	0.11	61.85
BKZ 4 c	0.07	21.42	0.15	74.54	0.02	20.30	0.15	73.31
BKZ 5 a	0.12	16.28	0.12	59.29	0.13	16.03	0.13	56.04
BKZ 5 b	0.04	20.84	0.04	67.65	0.04	21.14	0.03	67.88
BKZ 5 c	0.22	26.36	0.18	96.58	0.25	27.54	0.25	94.12
BKZ 6 a	1.07	60.59	0.81	186.54	1.07	63.34	1.01	180.00
BKZ 6 b	0.74	55.98	0.74	102.64	0.74	56.91	0.74	104.57
BKZ 6 c	0.31	85.32	0.31	298.78	0.31	86.31	0.31	292.67
Average	0.36	37.55	0.31	115.15	0.41	38.12	0.38	112.74

Table 6.2: Comparison among the Path Relinking strategies to the set BKZ.

observed that in twelve instances, at least one percentage deviation is negative. Thus, the results displayed in these tables indicate that the PR strategies may be used to improve the exact algorithms.

The third experiment evaluates the heuristics AMU Kasperski and Zieliński [2006], SBA, PR-R-Any, LPH and PM for the set of 75 theoretical instances. The results are reported in Table 6.3. The first column present the instance set's name while the second reports the average percentage deviation $\frac{\rho(X^{\text{B\&C}})-\rho(X^{\text{AMU}})}{\rho(X^{\text{B\&C}})}(\%)$ of the solutions provided by each set in AMU relative to those of B&C while the third column shows the average running time for AMU. The same data is reported for SBA, PR-R-Any, LPH and PM, respectively, in columns 4 and 5, 6 and 7, 8 and 9 and 10 and 11 . Again, best average solutions found for each instance set are highlighted and a negative percentage deviation means that the heuristic found a better feasible solution than a 900 second run of the exact algorithm. Results indicate that LPH is the heuristic that performed better on average while AMU returned the best average running times. It can be also observed that SBA found solutions as good as those of PR-R-Any and PM consuming much less computational time. The performance of SBA shows that an extensive scenario search does not necessarily result in better solutions for the *min-max regret* WSCP, since SBA deals with much less scenarios than PR-R-Any and PM.

Figure 6.2 shows a graphic comparing the solution improvement found by AMU, SBA, PR-R-Best, LPH and PM for the *min-max regret* WSCP over the time, in seconds, needed to find each new best solution for the instance scp43-2-1000. This graph is truncated after 140 seconds, because the slowest heuristic, LPH, stops after almost 136 seconds. It can be seen that LPH took nearly 90 seconds to find its best solution for this instance, despite it took only 20 seconds, approximately, to find a solution which is better than the ones found by the other heuristics. Finally, it can be noticed that AMU is the fastest heuristic, since it took only 0.01 second to return its best solution.

The fourth experiment evaluates the exact algorithms SBA+EB, SBA+B&C, PR+EB and PR+B&C. The results are reported in Table 6.4. The running times were also limited to 3600 seconds. The first and the second columns present, respectively, the name and the size ($|G|$) of each instance set. The third column reports the number of optimal solutions ($|O|$) found by SBA+EB in each instance set while the average relative optimality GAP ($\frac{\rho(X^{\text{BLD}})-z}{\rho(X^{\text{BLD}})}$)(%) of SBA+EB is reported in the fourth column. The fifth column shows SBA+EB average running time. The same data is reported for SBA+B&C, PR+EB and PR+B&C, respectively, in columns 6 to 8, 9 to 11 and 12 to 14. In each line, the algorithm that performed better and the one that found the best average gapes are highlighted. It can be observed that SBA+B&C performed better among these algorithms, because it found the optimal solution in thirteen instances,

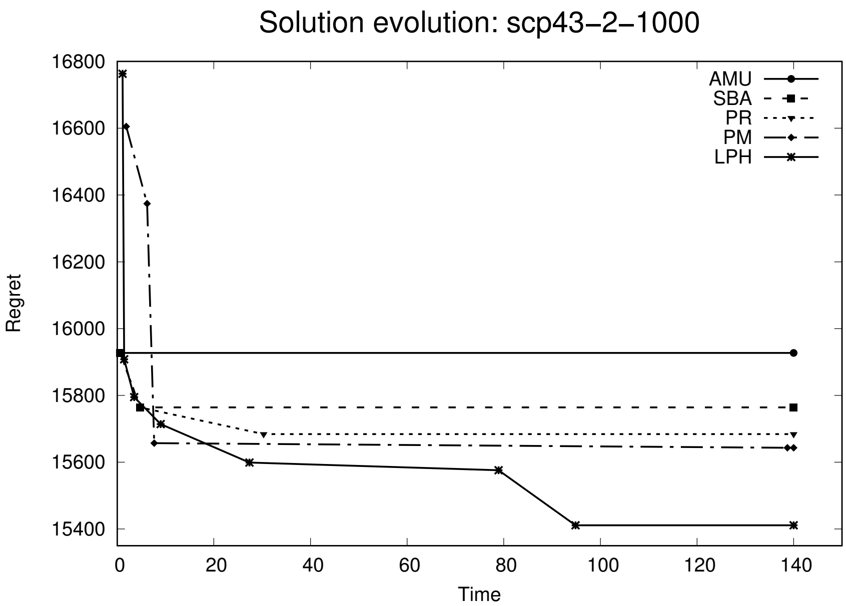

Figure 6.2: Solution improvement of AMU, SBA, PR-R-Best, LPH and PM to the *min-max regret* WSCP versus time (in seconds) needed to find each new best solution for the instance scp43-2-1000.

Instance	AMU		SBA		PR R Any		LPH		PM	
	Dev (%)	T (s)	Dev (%)	T(s)	Dev (%)	T (s)	Dev (%)	T (s)	Dev (%)	T (s)
BKZ 4 a	2.49	2.48	1.12	12.59	0.86	85.97	0.21	118.48	0.82	107.05
BKZ 4 b	1.15	4.18	0.15	23.72	0.12	64.34	0.55	270.29	0.10	183.14
BKZ 4 c	1.24	2.43	0.01	12.66	0.15	74.54	0.62	137.46	0.14	108.54
BKZ 5 a	0.37	2.21	0.13	10.44	0.12	59.29	0.07	44.82	0.49	157.47
BKZ 5 b	0.33	2.85	0.04	15.56	0.04	67.65	0.21	73.96	0.10	206.12
BKZ 5 c	1.09	3.08	0.25	15.76	0.18	96.58	0.23	140.24	0.16	277.15
BKZ 6 a	1.00	11.87	1.07	48.99	0.81	186.54	0.14	123.69	1.12	633.95
BKZ 6 b	0.71	19.40	0.74	49.96	0.74	102.64	0.05	383.51	0.81	742.57
BKZ 6 c	1.23	17.12	0.31	56.79	0.31	298.78	0.07	273.73	0.04	781.95
Average	1.07	7.29	0.43	27.38	0.31	115.15	0.19	174.02	0.42	355.33

Table 6.3: Comparison among the proposed heuristics to the set BKZ.

Instance	$\|G\|$	SBA+EB			SBA+B&C			PR+EB			PR+B&C		
		$\|O\|$	Gap	T (s)	$\|O\|$	Gap	T (s)	$\|O\|$	Gap	T (s)	$\|O\|$	Gap	T (s)
BKZ 4 a	10	0	11.65	900.00	0	12.55	900.00	0	12.72	900.00	0	12.77	900.00
BKZ 4 b	10	0	17.02	900.00	0	18.52	900.00	0	16.73	900.00	0	18.16	900.00
BKZ 4 c	10	0	13.11	900.00	0	14.89	900.00	0	13.61	900.00	0	14.73	900.00
BKZ 5 a	10	0	5.98	900.00	0	6.61	900.00	0	5.54	900.00	0	6.49	900.00
BKZ 5 b	10	0	7.74	900.00	0	8.44	900.00	0	7.36	900.00	0	8.18	900.00
BKZ 5 c	10	0	7.63	900.00	1	8.23	831.48	1	7.39	865.74	1	8.10	827.77
BKZ 6 a	5	3	1.56	688.04	5	0.00	384.46	3	1.06	680.79	3	1.15	518.25
BKZ 6 b	5	1	4.56	852.96	3	2.77	655.70	1	3.47	786.39	2	3.74	634.03
BKZ 6 c	5	1	3.17	891.93	4	0.47	604.62	1	1.84	887.45	4	0.67	607.03
Average	75	5	8.05	870.33	13	8.05	775.14	6	7.75	857.82	10	8.22	787.45

Table 6.4: Comparison among the SBA+EB, SBA+B&C, PR+EB and PR+B&C to the set BKZ.

while PR+B&C, PR+EB and SBA+EB encountered the optimal solution in ten, six and five instances, respectively. It can also be observed that coupling SBA or PR to the exact algorithms proposed by Pereira and Averbakh [2013] slightly improved the performance of the latter, since SBA+EB and PR+EB found, respectively, optimal solutions in five and six instances while EB solved only four instances while SBA+B&C and PR+B&C found, respectively, optimal solutions in thirteen and ten instances while B&C solved only eight instances. This happens due to the smaller number of iterations needed to found the optimal solution when AMU is replaced by SBA or PR, in spite of increasing the running time per iteration. This is an expected behavior since Γ^1 is usually larger when SBA is used while PR founds a large number of solutions at each iteration.

6.2 Results for the min-max regret MCLP

Let $w^l = \sum_{j \in M} l_j$ be an upper bound on the number of wounded inhabitants and $\psi^l(X^{\text{MMI}}) = \sum_{j \in X^{\text{MMI}}} l_j$ is the population having access to field hospitals in the optimal solution X^{MMI} for the *max-min lower scenario* MCLP. Moreover, let $w^u = \sum_{j \in M} u_j$ be an upper bound on the number of wounded inhabitants and $\psi^u(X^{\text{MMA}}) = \sum_{j \in X^{\text{MMA}}} u_j$ is the population having access to field hospitals in the optimal solution X^{MMA} for the *max-min upper scenario* MCLP. The following indicators are used in the table of results: $dev(w^u)$ and $dev(w^l)$ which stand respectively for the percentage $\psi^u(X^{\text{MMA}})/w^u(\%)$ and $\psi^l(X^{\text{MMI}})/w^l(\%)$.

The first experiment assesses the performance of the formulations of the *max-min upper scenario* MCLP and of the *max-min lower scenario* MCLP for the set of 100 Kathmandu instances. The results are reported in Table 6.5. The first column presents the instance name. The second column gives the upper bound w^u and the performance indicator $dev(w^u)$ values is given for each value of $T = \{2, ..., 6\}$ from columns 3 to 7. Column 8 presents w^l and the performance indicator $dev(w^l)$ values is given for each value of $T = \{2, ..., 6\}$ from columns 9 to 13. Optimal solutions for all instances of the *max-min upper scenario* MCLP and the *max-min lower scenario* MCLP were found in less than 0.1 seconds, which shows that these models can efficiently solve all Kathmandu instances using available ILP solver. Thus, there is no major obstacle in terms of running time to optimizing the installation of field hospitals in case of disaster using these models, proving the data is available. It can also be seen that for each additional field hospital available, the number of people accessing field hospitals grows around 10% on average, despite the scenario that is being optimized. Obviously, the number of field hospitals cannot be increased indefinitely due to the physical and human resources constraints.

The second experiment assesses the performance of the exact algorithms BLD, EB and B&C for the set of 100 realistic Kathmandu instances. The results are reported in Table 6.6. The first column presents the instance name. For each value of $T = \{2, ..., 6\}$, the following results are depicted: the maximum regret $\rho(X^*) = \psi^{s(X^*)}(Y^{s(X^*)}) - \psi^{s(X^*)}(X^*)$ of the *min-max regret* MCLP optimal solution X^*, and the *relative regret* $\varepsilon(X^*) = \frac{\rho(X^*)}{\psi^{s(X^*)}(Y^{s(X^*)})}$ of X^*, i.e. X^* is at most $\varepsilon(X^*)(\%)$ far from the best possible solution at all scenarios in S.

An important issue shown in the results is that $\rho(X^*)$ and $\varepsilon(X^*)$ values increase until $T = 3$ and then, they decrease progressively from $T = 4$ to $T = 6$. Let us consider T' as the cutting point, which determines if the number of field hospitals is greater than T', $\rho(X^*)$ and $\varepsilon(X^*)$ decreases. This happens since from $T = 2$ to $T = 3$, the number of inhabitants having access to a field hospital, independently of the scenario, grows slower than the number of inhabitants accessing a field hospital only in a subset of scenarios. This situation is reversed whenever more fields hospitals are deployed, i.e. from $T = 4$ to $T = 6$. Thus, the number of inhabitants having access to a field hospital, independently of the scenario, grows faster than this value for a subset of scenario. This provides the following insight for decision-makers. Whenever possible, it may be relevant to set a number of field hospitals greater than T'.

Both BLD, EB and B&C found similar optimal solutions for all Kathmandu instances. More than that, it can be observed that the *max-min upper scenario* MCLP, the *max-min lower scenario* MCLP and the *min-max regret* MCLP, despite the evaluation of the solutions which differs due to the optimization criteria, found similar solutions for Kathmandu instances, i.e. similar location to install the field hospitals. In terms of running time, all instances where solved in less than 0.1 seconds by BLD, EB and B&C.

The robust optimization models can provide solutions in a decision-making process in order to define priorities for locations concerning the installation of a predefined number of field hospitals, according to the optimization criteria the *max-min upper scenario* MCLP, the *max-min lower scenario* MCLP and the *min-max regret* MCLP. The context will define the target of optimization and in any case, the solutions are provided to support decision, not to replace the human decisions.

The third experiment compares the *max-min upper scenario* MCLP and the *max-min lower scenario* MCLP on the set of 225 theoretical instances. The running times were limited to 3600 seconds. The results are reported in Table 6.7. The first column

Instance	*Max-min upper scenario* MCLP						*Max-min lower scenario* MCLP					
	w^u	$dev(w^u)$					w^l	$dev(w^l)$				
		$T=2$	$T=3$	$T=4$	$T=5$	$T=6$		$T=2$	$T=3$	$T=4$	$T=5$	$T=6$
Kat-11-10-10	4652	23.15	36.31	47.83	56.81	65.37	3808	23.14	36.29	47.82	56.80	65.34
Kat-11-10-30	5500	23.14	36.31	47.84	56.82	65.36	2960	23.14	36.28	47.80	56.79	65.34
Kat-11-20-10	7803	24.03	37.99	48.66	58.69	68.32	6381	24.02	37.99	48.68	58.71	68.33
Kat-11-20-30	9220	24.03	37.98	48.67	58.70	68.32	4964	24.03	37.99	48.67	58.70	68.33
Kat-11-30-10	11514	24.43	38.42	49.36	59.27	69.01	9420	24.43	38.41	49.35	59.27	69.01
Kat-11-30-30	13609	24.43	38.41	49.35	59.26	69.00	7325	24.44	38.42	49.37	59.28	69.02
Kat-11-40-10	14524	24.59	38.73	49.83	59.83	69.69	11880	24.59	38.74	49.85	59.85	69.70
Kat-11-40-30	17164	24.59	38.74	49.84	59.85	69.67	9239	24.59	38.74	49.84	59.84	69.68
Kat-11-50-10	18016	24.92	39.21	50.42	60.54	70.63	14739	24.93	39.22	50.42	60.55	70.64
Kat-11-50-30	21293	24.92	39.21	50.41	60.53	70.63	11461	24.93	39.22	50.43	60.55	70.65
Kat-21-10-10	10259	20.30	30.89	41.14	50.87	60.06	8394	20.28	30.88	41.12	50.87	60.05
Kat-21-10-30	12126	20.28	30.89	41.12	50.86	60.05	6527	20.29	30.89	41.14	50.88	60.06
Kat-21-20-10	17100	21.15	31.74	41.98	52.08	61.62	13991	21.15	31.74	41.97	52.07	61.61
Kat-21-20-30	20213	21.14	31.74	41.97	52.07	61.62	10878	21.15	31.74	41.97	52.08	61.62
Kat-21-30-10	24818	21.58	32.44	42.73	52.77	62.69	20306	21.57	32.43	42.73	52.77	62.69
Kat-21-30-30	29332	21.57	32.43	42.73	52.76	62.68	15792	21.58	32.44	42.74	52.78	62.70
Kat-21-40-10	31796	21.88	32.91	43.43	53.63	63.67	26015	21.88	32.91	43.42	53.63	63.67
Kat-21-40-30	37579	21.87	32.92	43.43	53.62	63.67	20232	21.88	32.92	43.43	53.63	63.68
Kat-21-50-10	39264	21.84	32.79	43.19	53.43	63.40	32123	21.84	32.79	43.19	53.43	63.40
Kat-21-50-30	46404	21.85	32.80	43.19	53.43	63.40	24983	21.84	32.79	43.19	53.43	63.40
Average		22.78	35.14	45.86	55.79	65.44		22.79	35.14	45.86	55.80	65.45

Table 6.5: The *max-min upper scenario* MCLP and the *max-min lower scenario* MCLP for the set Kathmandu of instances.

	$T = 2$		$T = 3$		$T = 4$		$T = 5$		$T = 6$	
Instance	$\rho(X^*)$	$\varepsilon(X^*)$	$\rho(X^*)$	$\varepsilon(X^*)$	$\rho(X^*)$	$\varepsilon(X^*)$	$\rho(X^*)$	$\varepsilon(X^*)$	$\rho(X^*)$	$\varepsilon(X^*)$
Kat-11-10-10	34	3.72	53	3.69	19	1.03	56	2.52	0	0.00
Kat-11-10-30	244	26.26	524	32.79	328	18.82	271	13.88	223	10.34
Kat-11-20-10	3	0.20	83	3.31	149	4.58	111	2.88	0	0.00
Kat-11-20-30	339	22.13	911	32.57	729	23.42	407	12.26	107	3.06
Kat-11-30-10	0	0.00	102	2.74	193	3.99	188	3.26	0	0.00
Kat-11-30-30	507	22.07	1350	32.42	1065	22.75	627	12.62	119	2.30
Kat-11-40-10	11	0.38	145	3.05	244	3.96	242	3.29	0	0.00
Kat-11-40-30	661	22.54	1735	32.65	1355	22.73	766	12.17	111	1.69
Kat-11-50-10	22	0.60	168	2.82	313	4.04	327	3.53	0	0.00
Kat-11-50-30	833	22.57	2196	32.82	1716	22.89	990	12.48	62	0.76
Kat-21-10-10	150	8.10	311	10.71	289	7.73	150	3.39	0	0.00
Kat-21-10-30	642	32.66	1192	37.16	1259	31.92	928	21.84	583	12.95
Kat-21-20-10	252	7.85	520	10.48	549	8.55	252	3.34	0	0.00
Kat-21-20-30	1026	30.84	1942	36.00	2087	31.37	1453	20.41	737	9.91
Kat-21-30-10	386	8.10	736	10.05	788	8.33	422	3.79	0	0.00
Kat-21-30-30	1380	28.82	2684	34.38	3025	30.95	2112	20.33	976	8.97
Kat-21-40-10	518	8.34	989	10.35	1026	8.33	539	3.72	0	0.00
Kat-21-40-30	1704	27.79	3421	33.94	3789	30.13	2474	18.57	983	7.09
Kat-21-50-10	625	8.18	1192	10.17	1306	8.60	625	3.51	0	0.00
Kat-21-50-30	2192	28.66	4283	34.34	4727	30.47	3183	19.38	1391	8.07
Average		15.49		20.32		16.23		9.86		3.26

Table 6.6: Results of BLD, EB and B&C for the set Kathmandu of instances.

displays the constant $T < |M|$, which represents the maximum number of columns allowed in solution X, while the second one presents the name of each instance set. The average values of w^u, $\psi^u(X^{\text{MMA}})$ and $dev(w^u)$, previously defined, are displayed respectively in columns 3, 4 and 5. The average running time spent to find a the *max-min upper scenario* MCLP optimal solution is shown in the sixth column. Similar data is reported for the *max-min lower scenario* MCLP in columns 7 to 10, respectively. It can be seen that optimal solutions for both the *max-min upper scenario* MCLP and the *max-min lower scenario* MCLP were found efficiently (within 1 second) as for the realistic instances. Besides, one can observe that the density of the instances has little impact in the algorithm's running time. Moreover, the average relative number of people having access to field hospitals in the *max-min upper scenario* MCLP optimal solution is close to the *max-min lower scenario* MCLP optimal solution for all values of T.

The fourth experiment evaluates the exact algorithms BLD, EB and B&C on the set of 225 theoretical instances. The running times were limited to 3600 seconds. The results are reported in Table 6.8. The first column displays the values of constant T, which represents the maximum number of columns allowed in solution X, while the second one presents the name of each instance set. The average relative optimality gap $(\frac{\rho(X^{\text{BLD}})-z}{\rho(X^{\text{BLD}})})(\%)$ of BLD is reported in the third column, where z is the lower bound obtained by solving at optimality the master problem in the last iteration of BLD. The fourth column shows the average running time of BLD for each set, while the fifth column displays the number of cuts added to the master problem. The same data is reported for EB in columns 6 to 8 and for B&C in columns 9 to 11, respectively. It is worth mentioning that, in B&C, the lower bound z is obtained by the linear relaxation of formulation (4.12)-(4.14), (4.23), (4.25) and (5.2) with Δ^h containing one cut for each integer solution found by B&C.

In each line of Table 6.8, the algorithm that found the best average gaps is highlighted. It can be seen that the gaps, on average, were above 62% for $T = 0.1 \times |M|$, above 54% for $T = 0.2 \times |M|$ and above 53% for $T = 0.3 \times |M|$. Unlike other problems in the literature that used similar algorithms [Furini et al., 2015; Montemanni et al., 2007; Pereira and Averbakh, 2013], BLD, EB and B&C showed gaps above 60% on average, because most cuts did not significantly improve the formulation's linear relaxation. It can be also observed that B&C performed slightly better, on average, for $T = 0.1 \times |M|$ while BLD was the best exact algorithm, on average, for $T = 0.2 \times |M|$ and $T = 0.3 \times |M|$. This behavior probably occurs because of the excessive number of constraints (5.2) unnecessarily generated by EB and B&C which results in a formulation that is larger and harder to solve.

T	Instance	*Max min upper scenario* MCLP				*Max min* MCLP			
		w^u	$\psi^u(X^{\text{MMA}})$	$dev(w^u)$	T(s)	w^l	$\psi^l(X^{\text{MMI}})$	$dev(w^l)$	T(s)
$0.1 \times \lvert M \rvert$	BKZ 4	1000316.60	167439.27	16.74	0.01	502166.17	93435.47	18.61	0.01
	BKZ 5	1993439.17	176342.87	8.85	0.02	996868.87	96630.47	9.69	0.01
	BKZ 6	999444.93	170007.47	17.01	0.01	500900.27	94901.13	18.95	0.01
$0.2 \times \lvert M \rvert$	BKZ 4	1000316.60	315633.43	31.55	0.01	502166.17	179482.33	35.75	0.01
	BKZ 5	1993439.17	339179.07	17.02	0.02	996868.87	189474.27	19.01	0.01
	BKZ 6	999444.93	315548.93	31.57	0.01	500900.27	180032.87	35.95	0.01
$0.3 \times \lvert M \rvert$	BKZ 4	1000316.60	446306.20	44.56	0.01	502917.50	255542.20	50.82	0.01
	BKZ 5	1993439.17	488712.35	24.58	0.01	993280.40	276611.65	27.85	0.01
	BKZ 6	999444.93	445209.20	44.50	0.01	500777.00	256434.60	51.22	0.01
Average		1331066.90	318264.31	26.26	0.01	666316.17	180282.78	29.76	0.01

Table 6.7: Results of the *max-min upper scenario* MCLP and the *max-min lower scenario* MCLP for the set BKZ.

T	Instance	BLD			EB			B&C		
		Gap	T (s)	Cuts	Gap	T (s)	Cuts	Gap	T (s)	Cuts
$0.1 \times \lvert M \rvert$	BKZ 4	64.89	3600.00	8.00	64.65	3600.00	137.57	62.73	3600.00	4454.37
	BKZ 5	71.42	3600.00	7.30	70.71	3600.00	142.17	68.67	3600.00	5137.77
	BKZ 6	64.52	3600.00	7.20	65.03	3600.00	140.93	67.26	3600.00	5233.67
$0.2 \times \lvert M \rvert$	BKZ 4	55.47	3600.00	7.13	56.30	3600.00	116.27	64.52	3600.00	1494.30
	BKZ 5	64.31	3600.00	7.20	65.34	3600.00	101.57	73.61	3600.00	3047.10
	BKZ 6	54.47	3600.00	6.53	55.74	3600.00	98.73	63.96	3600.00	3000.83
$0.3 \times \lvert M \rvert$	BKZ 4	53.78	3600.00	6.70	54.66	3600.00	102.30	60.43	3600.00	765.97
	BKZ 5	57.87	3600.00	7.50	59.25	3600.00	115.77	68.16	3600.00	2556.40
	BKZ 6	53.31	3600.00	6.60	54.32	3600.00	102.40	60.70	3600.00	1582.27
Average		60.00	3600.00	7.13	60.67	3600.00	117.52	65.56	3600.00	3030.30

Table 6.8: Comparison among the exact algorithms to the set BKZ for the *min-max regret* MCLP.

The fifth experiment compares the four proposed Path Relinking strategies for the set of 225 theoretical instances. The results are reported in Table 6.9. The first column displays the constant $T < |M|$ while the second one presents the name of each instance set. The third column reports the average percentage deviation $\frac{\rho(X^{\text{B\&C}})-\rho(X^{\text{PRRBEST}})}{\rho(X^{\text{B\&C}})}(\%)$ of the solutions provided by PR-R-Best relative to those of B&C while the fourth column shows PR-R-Best average running time for each set. The same data is reported for PR-R-Any, PR-I-Best and PR-I-Any, respectively, in columns 5 and 6, 7 and 8 and 9-10. The best average solutions found for each instance set are highlighted and a negative percentage deviation means that the heuristic found a better feasible solution than a 3600 second run of the exact algorithm. Results indicate that PR-I-Any returned, on average, better solutions for $T = 0.1 \times |M|$, PR-I-Any, PR-R-Best and PR-I-Best returned, on average, similar solutions for $T = 0.2 \times |M|$ and, for $T = 0.3 \times |M|$, all strategies found similar solutions to all instances. In addition, as expected, PR-R-Best and PR-I-Best average running times are smaller than those of the other two strategies, because the number of solutions evaluated during their run is also smaller.

The sixth experiment evaluates the heuristics AMU Kasperski and Zieliński [2006], SBA, PR-I-Any, LPH and PM for the set of 225 theoretical instances. The results are reported in Table 6.10. The first column displays the values of constant T while the second one presents the name of each instance set. The third column reports the average percentage deviation $\frac{\rho(X^{\text{B\&C}})-\rho(X^{\text{AMU}})}{\rho(X^{\text{B\&C}})}(\%)$ of the solutions provided by AMU relative to those of B&C while the fourth column shows the AMU's average running time for each instance set. The same data is reported for SBA in columns 5 and 6, for PR in columns 7 and 8, for LPH in columns 9 and 10 and for PM in columns 11 and 12, respectively. The best average solutions found for each instance set are highlighted and a negative percentage deviation means that the heuristic found a better feasible solution than a 3600-second run of B&C. Results indicate that PR performed better on average for $T = 0.1 \times |M|$ and AMU, SBA, PR found similar solutions, on average, in the instances with $T = 0.2 \times |M|$ and $T = 0.3 \times |M|$ while, AMU returned the best average results in terms of running times. It is worth mentioning that AMU and SBA found solutions as good as PR and PM consuming much less computational time. This behavior means that an extensive scenario sampling does not necessarily result in better solutions for the *min-max regret* MCLP, since AMU and SBA explore much fewer scenarios than PR and PM in all instances. Moreover, it can be observed that LPH, which is the best known heuristic for the *min-max regret* WSCP Assunção et al. [2017] and the *min-max regret* Restricted Shortest Path problem Assunção et al. [2017], had the worse average results for the *min-max regret* MCLP, because the linear relaxation of mathematical formulation (5.26)-(5.28), (5.34)-(5.36) and (5.37) is weak.

T	Instance	PR R Best		PR R Any		PR I Best		PR I Any	
		Dev (%)	T (s)	Dev (%)	T(s)	Dev (%)	T (s)	Dev (%)	T (s)
$0.1 \times \|M\|$	BKZ 4	0.07	19.16	0.10	299.51	0.11	15.09	0.15	337.56
	BKZ 5	0.10	42.70	0.15	609.80	0.12	33.70	0.18	553.90
	BKZ 6	0.01	20.05	0.02	432.77	0.00	19.98	0.00	521.07
$0.2 \times \|M\|$	BKZ 4	0.04	63.01	0.04	3600.00	0.04	64.01	0.04	3600.00
	BKZ 5	0.02	137.60	0.03	3600.00	0.03	138.96	0.03	3600.00
	BKZ 6	1.39	112.57	1.39	3600.00	1.39	112.94	1.39	3600.00
$0.3 \times \|M\|$	BKZ 4	0.00	114.54	0.00	3600.00	0.05	129.49	0.00	3600.00
	BKZ 5	0.00	295.58	0.00	3600.00	0.00	393.82	0.00	3600.00
	BKZ 6	0.00	156.94	0.00	3600.00	0.00	219.69	0.00	3600.00
Average		0.13	106.90	0.12	2549.12	0.12	125.30	0.11	2556.95

Table 6.9: Comparison among the Path Relinking strategies to the set BKZ.

T	Instance	AMU		SBA		PR I Any		LPH		PM	
		Dev (%)	T (s)	Dev (%)	T(s)	Dev (%)	T (s)	Dev (%)	T (s)	Dev (%)	T (s)
$0.1 \times \|M\|$	BKZ 4	0.00	2.15	0.00	2.15	0.15	337.56	0.06	3183.70	0.94	15.14
	BKZ 5	0.00	3.99	0.00	4.00	0.18	605.30	0.26	3600.00	0.71	50.20
	BKZ 6	0.00	2.78	0.00	2.79	0.00	528.57	2.60	60.18	1.23	22.12
$0.2 \times \|M\|$	BKZ 4	0.00	0.07	0.02	2.63	0.04	3600.00	14.90	11.26	2.38	20.28
	BKZ 5	0.00	0.13	0.02	4.88	0.03	3600.00	31.84	253.29	1.14	68.15
	BKZ 6	1.40	0.09	1.39	3.34	1.39	3600.00	17.42	7.95	3.97	27.13
$0.3 \times \|M\|$	BKZ 4	0.00	0.11	0.00	4.13	0.05	3600.00	11.69	151.70	4.60	25.70
	BKZ 5	0.00	5.95	0.00	5.96	0.00	3600.00	5.17	253.29	1.96	84.64
	BKZ 6	0.00	3.94	0.00	3.95	0.00	3600.00	12.58	7.95	4.53	32.96
Average		0.16	2.14	0.15	3.76	0.11	2563.49	10.67	836.59	2.38	38.48

Table 6.10: Comparison among the proposed heuristics to the set BKZ.

Figure 6.3 shows a graphic comparing the solution improvement found by AMU, SBA, PR-R-Best, LPH and PM for the *min-max regret* MCLP over the time, in seconds, needed to find each new best solution for the instance scp41-2-1000 with $T = 0.1 \times |M|$. This graph is truncated after 3600 seconds, because it is the running time limit for all algorithms and this graphic is in log scale due to the huge time line. It can be seen that PR-I-Any took nearly 90 seconds to find the best solution for this instance while LPH found the worst best solution among all algorithms after almost 3600 seconds. Finally, it can be noticed that AMU is the fastest heuristic, since it took only 0.01 second to return its best solution.

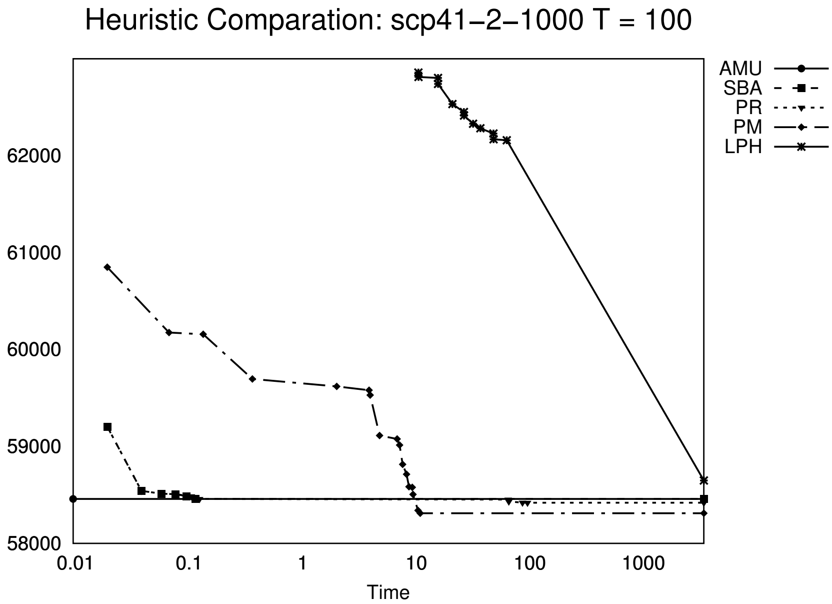

Figure 6.3: Solution improvement of AMU, SBA, PR-R-Best, LPH and PM to the *min-max regret* MCLP versus time (in seconds) needed to find each new best solution for the instance scp41-2-1000 with $T = 0.1 \times |M|$.

Chapter 7

Conclusions and future works

Two robust covering problems were dealt in this thesis: the *min-max regret* WSCP [Pereira and Averbakh, 2013] and the *min-max regret* MCLP. Despite both aim at finding the solution with the smallest maximum regret, the developed methods in Chapter 5 had different behaviors for the two problems. It means that methods which performed well to a *min-max regret* problem might not work properly to another due to factors such as the LP relaxation of its formulation and the way a solver deals with both the combinatorial optimization and the robust optimization counterparts.

Concerning the *min-max regret* WSCP, the exact algorithms proposed by Pereira and Averbakh [2013] solved only eight of the 75 instances while the exact algorithms coupled with SBA and PR, suggested on this work solved another six instances. It means that, despite SBA and PR slightly improve the performance of EB and B&C only 14 out of 75 instances for the *min-max regret* WSCP were solved. Therefore, there is still room for new exact methods since optimal solutions are not still known for several instances.

The four heuristics proposed to the *min-max regret* WSCP in this thesis returned better solutions, on average, than the AMU heuristic of Pereira and Averbakh [2013]. Moreover, the heuristics developed in this work have found better upper bounds than a 900-second run of exact algorithms in half of the instances, indicating that they might find near-optimal solutions. Finally, it is worth mentioning the heuristics LPH which found, on average, the better solutions for the *min-max regret* WSCP among the heuristics and, SBA which found solutions as good as the exact algorithms in a smaller amount of time.

The *min-max regret* MCLP was introduced in this thesis as a generalization of the classical MCLP. Besides, the proposed model found an application in disaster logistics, where field hospitals must be placed after large-scale emergencies such as earthquakes, hurricanes and floods. In this application, uncertainties are associated with the number of inhabitants affected after the disaster.

Realistic instances from of the earthquakes that hit Kathmandu, Nepal in 2015 were used in the experiments, as well as theoretical instances. The proposed robust optimization models provide different possibilities to get solutions and could be integrated in the decision-making process for post-disaster relief.

The numerical results indicates that the compact ILP formulations for *max-min upper scenario* MCLP and *max-min upper scenario* MCLP can be efficiently solved on both realistic and theoretical instances. Moreover, the exact algorithms BLD, EB and B&C efficiently solved the *min-max regret* MCLP on all 100 realistic instances. B&C may be considered a better algorithm to solve realistic instances of *min-max regret* MCLP, because it slightly improved the results of BLD and BE, which is the case of the real application of *min-max regret* MCLP, as the one focused in this thesis. Besides, B&C is faster and easier to implement than BLD and EB. In addition, the heuristics proposed for *min-max regret* MCLP found competitive results compared to the ones produced by B&C after one hour.

As future works, it is suggested to study the structures of the *min-max regret* WSCP and of the *min-max regret* MCLP formulations, to develop new cuts and valid inequalities for both problems, thus, obtaining better results for them. Moreover, it is also recommended to introduce a method which controls the cuts' growth in the exact algorithms, and to apply the proposed heuristics in other *min-max regret* problems.

Appendix A

Résumé étendu en français Problèmes de couverture robuste: formulations, algorithmes et application

Chapitre 1: Introduction

L'Optimisation Robuste (OR) est une méthodologie qui traite des problèmes sujets à des paramètres incertains, dont les possibilités de valeurs pour ces paramètres sont représentés par un ensemble déterministe de données [Aissi et al., 2009; Kasperski et al., 2005; Kouvelis and Yu, 1997]. L'OR est apparue à la fin des années soixante, appliquée à des problèmes en finance [Gupta and Rosenhead, 1968; Rosenhead et al., 1972]. Elle était généralement utilisée pour éviter les impacts indésirables dus à des approximations, des données incomplètes, imprécises ou des ambiguïtés. Un recueil bibliographique est trouvé en Roy [2010] couvrant les différentes définitions de la robustesse dans le domaine de la recherche opérationnelle. De plus, les revues bibliographiques Aissi et al. [2009]; Bertsimas et al. [2015]; Coco et al. [2014b]; Gabrel et al. [2014]; Kasperski and Zieliński [2016] abordent des stratégies de OR et des aspects théoriques. Les travaux Conde [2012]; Chassein and Goerigk [2015]; Kasperski and Zieliński [2006]; Montemanni et al. [2004, 2007]; Siddiqui et al. [2011] sont dédiés à des algorithmes exacts et heuristiques pour des problèmes d'OR.

La version robuste de plusieurs problèmes d'optimisation classiques a été traitée dans la littérature scientifique, tels que le Problème du plus court chemin robuste

[Coco et al., 2014a; Karasan et al., 2001], le Problème de l'arbre couvrant robuste [Pérez-Galarce et al., 2014; Yaman et al., 2001], le Problème d'affectation robuste [Pereira and Averbakh, 2011] et le Problème d'arbre de plus court chemins robuste [Carvalho et al., 2016]. Ces problèmes sont NP-difficiles [Aissi et al., 2009], bien que la version correspondante déterministe soit polynomiale. Les problèmes d'OR dont la version déterministe est déjà NP-difficile ont aussi donné lieu à des études scientifiques. Nous pouvons citer le problème du voyageur de commerce robuste [Montemanni et al., 2007], le problème des ensembles de couverture robuste [Pereira and Averbakh, 2013], le problème du sac-à-dos robuste [Furini et al., 2015], le problème du plus court chemin restreint robuste [Assunção et al., 2017] et le problème de tournées robustes de véhicules [S.-Charris et al., 2015, 2016]. Ces problèmes d'OR soulèvent plusieurs défis en termes d'algorithmes et de modèles mathématiques, car la plupart sont NP-difficile. De plus, les modèles mathématiques ont couramment un nombre exponentiel de contraintes, ce qui amène une difficulté supplémentaire pour leurs résolutions.

Cette thèse aborde le développement d'approches OR pour le problème de couverture. De plus, une application réelle en logistique de crise est aussi étudiée. Les données incertaines sont modelées par des intervalles continues de valeurs. Ainsi, la réalisation de chaque paramètre incertain correspond à un scénario qui peut se produit. L'objectif est de trouver une solution satisfaisante vis-à-vis de tous les scénarios, communément appelée solution robuste. Un des critères d'OR les plus utilisés est le *min-max regret*. Il a a été introduit par Wald [1939] dans le cadre de la théorie des jeux et ensuite adapté à l'OR par Yu and Yang [1997]. Il s'agit d'un critère moins conservative que le critère *min-max* [Neumann, 1928; Soyster, 1973].

On considère que les paramètres sujets à des incertitudes sont modélisés à partir d'un intervalle continue de données et le critère d'optimisation est le *min-max regret*. Deux problèmes sont considérés dans cette thèse: le Problème des ensembles de couverture robuste (*min-max regret* WSCP, de l'anglais *Min-max regret Weighted Set Covering Problem*) et le Problème de couverture maximale robuste (*min-max regret* MCLP, de l'anglais *Min-max regret Maximum Covering Location Problem*). Le *min-max regret* WSCP a été introduit par Pereira and Averbakh [2013] et est définit comme suit. Étant donné une matrice $\{a_{ij}\}$, l'ensemble N de lignes, l'ensemble M de colonnes et un intervalle de coût $[l_j, u_j]$ associé à chaque colonne $j \in M$, où $l_j \in \mathbb{N}$, $u_j \in \mathbb{N}$, et $l_j \leq u_j$, le *min-max regret*-WSCP vise à définir une solution $X^* \subseteq M$ qui minimise le regret maximal pour tous les scénarios, de façon à ce que chaque ligne N soit couverte par au moins une colonne de X. $T < |M|$ est une constante pour limiter le nombre de colonnes qui vont entrer dans la solution, où l'intervalle $[l_j, u_j]$ est associé à chaque colonne $j \in M$. Le *min-max regret* MCLP consiste à trouver une solution $X^* \subseteq M$ qui minimise le regret maximal en considérant tous les scénarios, tels que chaque ligne en N est couverte par au moins une colonne de X et $|X| \leq T$. Ce problème est motivé par une application traitée dans le projet OLIC [2015] et dédiée à l'optimisation pour la localisation et l'installation de tentes médicales lors des événements catastrophiques tels que le tremblement de terre qui a eu lieu à Katmandou au Népal en 2015. Dans cette application, les incertitudes sont associées aux nombre de personne (demande) ayant besoin de soins médicaux. À notre connaissance, ce travail de thèse est le premier à étudier le *min-max regret*. On considère $\{a_{ij}\}$, N, M, l_j et u_j comme défini précédemment.

Des méthodes exactes et heuristiques ont été développées et appliquées aux problèmes *min-max regret* WSCP et *min-max regret*-MCLP. Trois algorithmes exacts ont été développés: une décomposition de Benders [Montemanni et al., 2007], une décomposition de type Benders étendue proposée par Fischetti et al. [2010] et un algorithme de type Branch-and-Cut (B&C) [Mitchell, 2002]. De plus, des heuristiques basées sur des scénarios [Kasperski and Zieliński, 2006; Coco et al., 2015], un *Path Relinking* [Glover and Laguna, 1993], une Pilot Method [Voss et al., 2005] et des heuristiques basées sur la programmation linéaire [Dantzig, 1963] sont aussi proposés.

Ce document est organisé comme suit:

- Le Chapitre 2 introduit les définitions pour des problèmes lié à ceux ciblés dans cette thèse, puis pour le *min-max regret* WSCP et le *min-max regret* MCLP, en incluant la complexité et des exemples.

- Dans le Chapitre 3, une révision bibliographique des problèmes de couverture et des applications liées aux problèmes traités sont fournies. De plus, une vision globale analytique par rapport aux travaux existants et des contributions additionnelles est donnée.

- Le Chapitre 4 aborde des formulations de Programmation Linéaire en Nombres Entiers (PLNE) pour le *min-max regret* MCLP et trois modèles robustes pour le problème d'installation de tentes médicales : le *max-max* MCLP, le *min-max* MCLP et le *min-max regret* MCLP.

- Dans le Chapitre 5, des méthodes génériques sont proposées et appliquées pour le *min-max regret* MCLP et le *min-max regret* MCLP. Plus précisément, une décomposition de Benders, une décomposition de Benders étendue et un algorithme de type B&C, ainsi que cinq heuristiques sont présentées.

- Le Chapitre 6 est dédié aux expérimentations numériques pour les méthodes proposées, en utilisant des indicateurs de performance classiques tels que le gap, la déviation à l'optimum, les limites inférieures et supérieures et le temps de calcul.

- Dans le Chapitre 7, les principales contributions de cette thèse seront résumées, incluant modèles et méthodes. Enfin, des directions de recherche futures et les opportunités sont aussi abordées.

Chapitre 2: Problèmes de couverture robuste

Dans ce chapitre, initialement, les versions déterministes (sans incertitudes) liées aux problèmes *min-max regret* WSCP et *min-max regret* MCLP sont définies ci-après. Puis, les deux problèmes d'OR traités dans cette thèse, *i.e.* *min-max regret* WSCP et *min-max regret* MCLP seront détaillés.

Problèmes déterministes

Les problèmes de couverture est l'une des classe de problèmes les plus étudiées depuis les débuts de la recherche opérationnelle [Caprara et al., 2000; Farahani et al., 2012; Edmonds, 1962]. Le problème classique de couverture pondérée (WSCP, de l'anglais *Weighted Set Covering Problem*) a été introduit par Edmonds [1962] et est NP-difficile [Garey and Johnson, 1979]. On considère une matrice $\{a_{ij}\}$, l'ensemble N de lignes et l'ensemble M de colonnes et un coût $c_j \geq 0$ associé à chaque colonne $j \in M$. L'objectif du WSCP est de trouver un sous-ensemble $X \subseteq M$ tel que les coûts totaux sont minimum, de façon à ce que chaque ligne de N soit couverte par au moins une colonne de X.

Le Problème de couverture maximale (MCLP, de l'anglais *Maximal Covering Location Problem*) est une extension du WSCP, introduit par Church and Velle [1974], qui est NP-difficile [Garey and Johnson, 1979]. Il est défini par une matrice $\{a_{ij}\}$, où, N et M correspondent, respectivement, aux ensembles de lignes et de colonnes. Un profit $b_j \geq 0$ est associé à chaque colonne $j \in M$. Étant donné une constante $T < |M|$, MCLP consiste à définir un sous-ensemble $X \subseteq M$ de profit maximal, de façon à ce que $|X| \leq T$ et chaque ligne de N soit couverte par une colonne de X.

WSCP et MCLP trouvent des applications pratiques en planification [Caprara et al., 1999; Fisher and Rosenwein, 1989; Smith, 1988], en métallurgie [Vasko et al., 1989], en services d'urgences médicales [Brotcorne et al., 2003; Gendrau et al., 1997;

Li et al., 2011], en interventions post-catastrophes [Jia et al., 2007a,b; W.Yia and Özdamarb, 2007], en localisation de facilités [Farahani et al., 2012; Schilling et al., 1993], en localisation de facilités dans des sites préservés [Church et al., 1996; Snyder and Haight, 2016; Tong and Murray, 2009], en géographie [Murray, 2005], etc. La plupart de ces applications sont sujettes à des incertitudes sur les données, ce qui motive l'étude de tels problèmes dans le cadre de l'OR.

Le problème robuste WSCP

Le *min-max regret* WSCP proposé par Pereira and Averbakh [2013] est la version robuste du WSCP où les coûts associés aux colonnes sont incertains et modélisés par un intervalle de valeurs continues. On considère N, M et $\{a_{ij}\}$ comme définis précédemment et $[l_j, u_j]$ un intervalle représentant les coûts possibles pour chaque colonne $j \in M$. De plus, un scénario $s \in S$ est l'affectation d'une valeur unique $c_j^s \in [l_j, u_j]$ pour chaque colonne $j \in M$, où S est l'ensemble de toutes les combinaisons de valeurs possibles pour les coûts des colonnes de la matrice $\{a_{ij}\}$. Le *min-max regret* WSCP consiste à déterminer $X \subseteq M$ tel que chaque ligne en N est couverte par au moins une colonne de X. Cependant, le coût de chaque colonne est incertain. Ainsi, la fonction objectif doit pouvoir évaluer une solution selon le critère d'optimisation *min-max regret*. Ceci en considérant l'ensemble des valeurs possibles pour chaque colonne.

Étant donné Γ l'ensemble des solutions réalisables et $\omega^s(X) = \sum_{j \in X} c_j^s$ le coût d'une solution $X \in \Gamma$ pour le scénario $s \in S$ où c_j^s est le coût de la colonne $j \in M$ en s. Le *regret* $\rho^s(X)$ d'une solution $X \in \Gamma$ pour le scénario $s \in S$ est défini comme la différence entre $\omega^s(X)$ et $\omega^s(Y^s)$, où Y^s est la solution optimale pour le scénario s, *i.e.* le regret d'utiliser X au lieu de Y^s si le scénario s se produit. Le *min-max regret* WSCP a pour objectif de trouver une solution X^* minimisant le regret maximal (voir l'Équation (A.1)).

$$X^* = \arg \min_{X \in \Gamma} \max_{s \in S} \left\{ \omega^s(X) - \omega^s(Y^s) \right\} \tag{A.1}$$

Il est important de souligner qu'il y a un nombre infini de scénarios en S. Néanmoins, étant donné une solution $X \in \Gamma$, le scénario $s(X)$ de regret maximal pour X peut être calculé en temps polynomial, comme prouvé par Karasan et al. [2001]. Ainsi, Karasan et al. [2001] a obtenu un important résultat théorique: pour les problèmes d'OR avec critère d'optimisation min-max regret, dont la version déterministe est un problème de minimisation, le scénario $s(X)$ peut être calculé en temps polynomial. $s(X)$ est le scénario obtenu en fixant les valeurs $c_j^{s(X)} = u_j$, $\forall j \in X$ et $c_j^{s(X)} = l_j$, $\forall j \in M \setminus X$.

Le problème robuste MCLP

Le *min-max regret* MCLP proposé dans cette thèse est la version robuste du MCLP. Le profit de chaque colonne est incertain et modélisé par le biais d'intervalles de valeurs. N, M, $\{a_{ij}\}$ et T ont été déjà définis et $[l_j, u_j]$ est l'intervalle pour chaque colonne $j \in M$. Un scénario $s \in S$ est l'affectation d'une valeur unique $b_j^s \in [l_j, u_j]$ pour chaque colonne $j \in M$ où S est l'ensemble des possibles valeurs pour les profits associés aux colonnes. Le *min-max regret* MCLP consiste à déterminer $X \subseteq M$ tel que $|X| \leq T$ et chaque ligne en N est couverte par au moins une colonne de X.

On considère Δ l'ensemble des solutions réalisables et $\psi^s(X) = \sum_{j \in X} b_j^s$ le profit d'une solution $X \in \Delta$ pour chaque scénario $s \in S$, où b_j^s est le bénéfice de la colonne $j \in M$ dans s. Le *regret* d'une solution $X \in \Delta$ pour un scénario $s \in S$ est défini comme la différence entre $\psi^s(Y^s)$ et $\psi^s(X)$, où Y^s est la solution optimale pour le scénario s, *i.e.* le regret d'utiliser X au lieu de Y^s si le scénario s se produit. Le *min-max regret* MCLP a pour objectif de trouver une solution X^* qui minimise le regret maximal (voir l'Équation (A.2)).

$$X^* = \arg \min_{X \in \Gamma} \max_{s \in S} \left\{ \psi^s(Y^s) - \psi^s(X) \right\} \tag{A.2}$$

Les intervalles ont un nombre infini de scénarios dans S. Mais Furini et al. [2015] a montré qu'il n'est pas nécessaire d'évaluer toutes les solutions pour tous les scénarios. Étant donné une solution $X \in \Delta$, le scénario $s(X)$ où le regret de X est maximal peut être calculé en temps polynomial. Selon Furini et al. [2015], pour obtenir une solution robuste, il suffit de considérer le scénario $s(X)$ où les colonnes ont leurs valeurs fixées comme suit: $b_j^{s(X)} = l_j$, $\forall j \in X$ et $b_j^{s(X)} = u_j$, $\forall j \in M \setminus X$.

Le *min-max regret* MCLP peut modéliser le problème d'installation de tentes médicales après une catastrophe, tel que le tremblement de terre qui a eu lieu à Katmandou au Nepal en avril 2015. Dans ce problème, T tentes médicales doivent être installées sur un ensemble de M sites (colonnes), tout en couvrant l'ensemble Nles hôpitaux (lignes). L'objectif est de maximiser le nombre de personnes qui vont accéder aux tentes médicales. Ceci permettra de réaliser un tri et de ne pas transférer que les cas les plus graves aux hôpitaux. Les incertitudes sont associées au nombre de personnes blessées (demande). Ceci est un cas intéressant, où l'OR peut être utilisée. Pour l'application des tentes médicales, les scénarios sont définis en fonction d'un nombre probable de personnes affectées par la catastrophe dans une zone. Il est estimé à partir de la population de la zone, de la magnitude du tremblement de terre, des surfaces bâties affectées, les replis, etc. Cette application est apparue dans le cadre du projet OLIC [2015], dédié à l'optimisation d'interventions post-catastrophes majeures.

Chapitre 3: Révision bibliographique

Le WSCP, proposé par Edmonds [1962], est l'un des problèmes pionnier de recherche opérationnelle. Il a été classifié comme NP-difficile par [Garey and Johnson, 1979]. Le travail d'Edmonds [1962] a défini des résultats théoriques pour le WSCP et a proposé une formulation mathématique. Des recueils bibliographique pour des problèmes de couverture sont trouvés dans Balas [1983]; Beasley [1990b]; Caprara et al. [2000]; Ceria et al. [1997].

Le MCLP a été introduit par Church and Velle [1974] et appartient aussi à la classe des problèmes NP-difficiles [Garey and Johnson, 1979]. Church and Velle [1974] ont présenté une formulation, un algorithme de programmation linéaire et des heuristiques gloutonnes qui sont capable de résoudre des instances contenant 55 lignes et colonnes. Les travaux suivants sont des points d'entrés pour la version déterministe du MCLP : Abravaya and Segal [2010]; Farahani et al. [2012]; Karasakal and Karasakal [2004].

Quelques contributions dans la littérature ciblent les problèmes de couverture sujets à des paramètres incertains, tels que les travaux de [Beraldi and Ruszczyński, 2002; Pereira and Averbakh, 2013; Lutter et al., 2017]. Les paramètres incertains pour les problèmes de couverture sont notamment associés aux coûts des colonnes Pereira and Averbakh [2013], au budget uncertainty Lutter et al. [2017] ou encore à la probabilité de choisir une colonne [Beraldi and Ruszczyński, 2002].

Le travail pionnier pour le problème de couverture *min-max regret* est celui de Pereira and Averbakh [2013]. Les auteurs ont introduit une formulation linéaire, des méthodes exactes basées sur la décomposition de Benders, un B&C, un algorithme génétique et une heuristique hybride pour le *min-max regret* WSCP. Les résultats indiquent que le B&C a une meilleure performance parmi les méthodes exactes proposées et que l'heuristique hybride produit les meilleures limites supérieures. Dans cette thèse, les méthodes proposées par Pereira and Averbakh [2013] ont été reproduites et améliorées. De plus, d'autres méthodes sont proposées telles que des heuristiques déterministes et hybrides couplées aux algorithmes de Pereira and Averbakh [2013]. Le *min-max regret* MCLP est une extension du *min-max regret* WSCP et les modèles de PLNE et méthodes ont été adaptés en conséquence. Les contributions additionnelles en termes de méthodes sont détaillées dans le Chapitre 5.

Le Tableau A.1 résume les caractéristiques des problèmes et approches étudiées dans cette thèse et celles présents dans la littérature scientifique. La première colonne montre les références. Les trois prochaines colonnes identifient si la version du problème étudié est déterministe (DV de l'anglais, Deterministic Version) ou s'il y a des incertitudes sur les données (UV de l'anglais, Uncertain Version). Dans le cas UV, les méthodes ciblées sont aussi identifiées, c-à-d. OR et PS de Programmation stochastique. Les contraintes traitées dans les problèmes correspondants sont indiquées dans les trois colonnes qui suivent. La colonne "Couverture" indique si le problème a une contrainte de ce type, alors que les colonnes "Localisation" et "Sac-à-dos" précisent si ce type de sous-problème est considérée ou pas. Finalement, la dernière colonne indique si le travail a utilisé des Instances Réelles (IR) ou pas pour tester les méthodes abordées.

Les caractéristiques du *min-max regret* WSCP et *min-max regret* MCLP sont données dans les deux dernières lignes du Tableau 3.1. Notons que le *min-max regret* WSCP a été appliqué notamment dans le cadre théorique. À notre connaissance le *min-max regret* MCLP est le premier a intégrer à la fois les contraintes de couverture et de sac-à-dos tout en considérant des paramètres incertains. Le *min-max regret* MCLP a été évalué en utilisant des instances réelles obtenues pour le tremblement de terre qui a touché Katmandou au Népal en 2015.

Chapitre 4: Formulations mathématiques

Les formulations mathématiques pour le WSCP et pour le *min-max regret* WSCP sont décrites ci-dessous, suivies de la présentation des formulations mathématiques pour le MCLP et le *min-max regret* MCLP. Les modèles mathématiques pour le WSCP, le MCLP et le *min-max regret* WSCP, proposés respectivement par [Edmonds, 1962; Church and Velle, 1974; Pereira and Averbakh, 2013] sont révisés, tandis que la formulation pour le *min-max regret* MCLP est proposée dans cette thèse, inspirée de l'étude de Furini et al. [2015].

Formulation déterministe et robuste pour le WSCP

Soit l'ensemble N de lignes, l'ensemble M de colonnes et la matrice a_{ij}, où a_{ij} indique si la ligne $i \in N$ est couverte par au moins une colonne $j \in M$ ($a_{ij} = 1$), ou non ($a_{ij} = 0$). Le coût pour sélectionner une colonne est donné par c_j^s, $j \in M$ dans le scénario $s \in S$. Les formulations pour le WSCP et le *min-max regret* WSCP utilisent les variables de décisions $x \in \{0,1\}^{|M|}$, de sorte que $x_j = 1$ si la colonne $j \in M$ est sélectionnée, ou non $x_j = 0$. Afin de simplifier la notation, le sous-ensemble des colonnes $X \subseteq M$ et le vecteur caractéristique $|M|$-dimensionnel x de X sont référencé comme X.

La formulation WSCP, introduite par Edmonds [1962], est fournie par la fonction objectif (A.3) et les contraintes (A.4) et (A.5). L'objectif est de trouver une solution $X \subseteq M$ de coût minimal dans le scénario $s \in S$, où chaque ligne $i \in N$ est couverte par au moins une colonne $j \in x$. Les inégalités (A.4) assurent que chaque ligne de

Références	DV	UV		Couverture	Localisation	Sac-à-dos	IR
		OR	PS				
WSCP [Edmonds, 1962]	•			•			
MCLP [Church and Velle, 1974]	•			•		•	
WSPP [Garfinkel and Nemhauser, 1969]	•			•		•	
LSCP [Toregas et al., 1971]	•				•		
MCP [Nemhauser et al., 1978]	•					•	
PSCP [Beraldi and Ruszczyński, 2002]			•		•		
PSCP-FLP [Beraldi et al., 2004]			•		•		•
Budgeted Uncertainty KP [Bertsimas and Sim, 2004]		•				•	
Min-max regret FLP [Snyder, 2006]		•			•		
FLP-VP [Dessouky et al., 2006]	•				•		•
FLP-LSE [Jia et al., 2007a]	•			•	•		•
MC-LSE [Jia et al., 2007b]	•			•			•
CFL-HE [Horner and Downs, 2010]	•				•		•
LSECP [Huang et al., 2010]	•				•		•
Budgeted Uncertainty SCP [Lutter et al., 2017]		•	•	•			
Min-max regret KP [Furini et al., 2015]		•				•	
MCLP-EMS [Degel et al., 2015]	•			•	•	•	•
MPFLP-LSE [Duhamel et al., 2016]					•		•
Min-max regret WSCP [Pereira and Averbakh, 2013]		•		•			
Min-max regret MCLP		•		•		•	•

Table A.1: Vision globale des caractéristiques du *min-max regret* WSCP, du *min-max regret* MCLP et des problèmes traités dans la littérature.

N est couverte par au moins une colonne de M. De plus, le domaine des variables x est défini en (A.5). L'ensemble Γ de solutions réalisables est donné par les contraintes (A.4) et (A.5).

$$\min_{x\in\Gamma} \quad \sum_{j\in M} c_j^s x_j \tag{A.3}$$

$$s.t.$$

$$\sum_{j\in M} a_{ij}x_j \geq 1 \qquad \forall i \in N \tag{A.4}$$

$$x \in \{0,1\}^{|M|} \tag{A.5}$$

La formulation mathématique pour le *min-max regret* WSCP est donnée par la fonction objectif (A.6), qui minimise le regret maximal de x, les contraintes (A.7) et (A.8) qui assurent que $\theta = \omega^{s(x)}(y^{s(x)})$. Les contraintes (A.4) et (A.5) sont représentées par $x \in \Gamma$. Notons qu'il existe un nombre exponentiel de contraintes de type (A.7). Á notre connaissance, il n'existe pas dans la littérature de PLNE compacts pour les problèmes min-max regret, avec des données modélisées par des intervalles et dont la version déterministe est NP-difficile.

$$\min_{x\in\Gamma} \quad \Big\{\sum_{j\in M} u_j x_j - \theta\Big\} \tag{A.6}$$

$$s.t.$$

$$\theta \leq \sum_{j\in M} l_j y_j + \sum_{j\in M} y_j\,(u_j - l_j)\,x_j \qquad \forall y \in \Gamma \tag{A.7}$$

$$\theta \quad free \tag{A.8}$$

Le *min-max regret* WSCP est NP-difficile car calculer le coût pour une solution $X \in \theta$ implique résoudre une instance pour le WSCP dans le scénario $s(X)$. Ainsi, la version de décision du *min-max regret* WSCP est dans P si et seulement si $P = NP$.

Formulation déterministe et robuste pour le MCLP

Nous avons étudié trois critères d'optimisation robuste utilisés fréquemment dans la littérature scientifique. L'idée est de fournir un ensemble de modèles d'OR pour un problème trouvé dans le cadre des catastrophe majeures, en incluant des méthodes adaptées capables d'être déployées selon les caractéristiques de la catastrophe et les données disponibles. Le premier modèle correspond au *max-max* MCLP qui maximise le nombre de personnes à soigner dans les tentes médicales dans le scénario où le nombre de blessés est important. La logique de ce modèle repose sur le fait que l'optimisation dans le pire des cas doit fournir une solution acceptable pour l'ensemble des scénarios lorsque la demande dépasse largement la capacité d'accueil.

Le *max-min* MCLP maximise le nombre de personnes qui seront soignées dans les tentes médicales dans le scénario où le nombre de personnes cherchant de l'aide est minimal. Dans ce sens, la solution optimale pour le *max-min* MCLP indique le nombre de personnes minimal (dans la borne inférieure lb) qui auront accès aux tentes médicales, en considérant l'ensemble des scénarios.

Le *min-max regret* MCLP a été aussi développé pour le problème d'installation des tentes médicales, où la signification des paramètres et variables est la suivante: les ensembles N de lignes et M de colonnes représentent respectivement les hôpitaux et les endroits potentiels pour installer des tentes médicales. De plus, $T \in \mathbb{N}$ est une constante qui indique le nombre de tentes médicales disponibles et $\{a_{ij}\}$ est une matrice qui définit si un hôpital $i \in N$ est couvert par une tente médicale $j \in M$, donc $a_{ij} = 1$, sinon $a_{ij} = 0$. Le terme b_j^s est référé ici comme le profit obtenu si la colonne $j \in M$ dans le scénario $s \in S$ est sélectionnée.

Les formulations déterministes (sans incertitudes) et robustes pour le MCLP utilisent des variables de décision $x \in \{0,1\}^{|M|}$, indiquant si la colonne j est sélectionnée, donc $x_j = 1$, ou sinon $x_j = 0$. Comme mentionné précédemment et pour simplifier, le sous-ensemble $X \subseteq M$ et le vecteur caractéristique $|M|$-dimensionnel x de X sont référencés comme X.

Le MCLP déterministe

La formulation du MCLP, proposé par Church and Velle [1974], est donnée par la fonction objectif (A.9) et les contraintes de (A.10) à (A.12). L'objectif est de trouver une solution $|x| \leq T$ de profit maximal où chaque ligne $i \in N$ est couverte par au moins une colonne $j \in x$, aspect assuré par les inégalités (A.10). De plus, la contrainte (A.11) définit qu'au plus T colonnes seront sélectionnées. Finalement, les variables x sont définies en (A.12). L'ensemble Δ de solutions réalisables est donné par les contraintes de (A.10) à (A.12).

$$\max_{x \in \Delta} \quad \sum_{j \in M} b_j^s x_j \tag{A.9}$$

$$\sum_{j \in M} a_{ij} x_j \geq 1 \qquad \forall i \in N \tag{A.10}$$

$$\sum_{j \in M} x_j \leq T \tag{A.11}$$

$$x \in \{0,1\}^{|M|} \tag{A.12}$$

Max-Max MCLP

Max-max MCLP est définit comme suit. On considère la fonction objectif (A.13), ce modèle calcule le nombre de personnes ayant accès aux tentes médicales dans le scénario où le nombre de personnes cherchant de l'aide est maximal. Or, le scénario où le nombre de personnes est maximal est celui où $b_j^s = u_j$. Le modèle *max-max* MCLP ILP est défini par la fonction objectif (A.13) et les contraintes de (A.10) à (A.12).

$$\max_{x \in \Gamma} \sum_{j \in M} u_j x_j \qquad s.t. \tag{A.13}$$

Constraintes (A.10) à (A.12)

Max-Min MCLP

La fonction objectif (A.14) calcule le nombre de personnes cherchant de l'aide dans les tentes médicales, dans le scénario où le nombre de personnes blessées est minimal. Ceci correspond au scénario $b_j^s = l_j$. Le *max-min* MCLP ILP est donné par la fonction objectif (A.14) et les contraintes de (A.10) à (A.12).

$$\max_{x \in \Gamma} \sum_{j \in M} l_j x_j \qquad s.t. \tag{A.14}$$

Constraintes (A.10) to (A.12)

Toutes les instances pour le *max-min* MCLP sont équivalentes à une instance pour le MCLP où $b_j = l_j$ et peuvent être réduite en temps polynomial. Par conséquent, le *max-min* MCLP est NP-difficile et il est possible d'adapter et d'utiliser un algorithme pour MCLP afin de résoudre le *max-min* MCLP.

Min-max regret MCLP

La formulation MILP pour le *min-max regret* est définie par: la fonction objectif (A.15), qui minimise le regret maximal de x; les contraintes (4.24) et (A.17) qui assurent que $\mu = \psi^{s(x)}(y^{s(x)})$; et les contraintes de (A.10) à (A.12), définissant $x \in \Delta$. Il existe un nombre exponentiel de contraintes (A.16).

$$\min_{x \in \Delta} \quad \left\{ \mu - \sum_{j \in M} l_j x_j \right\} \tag{A.15}$$

$$s.t.$$

$$\mu \geq \sum_{j \in M} u_j y_j + \sum_{j \in M} y_j \left(l_j - u_j\right) x_j \qquad \forall y \in \Delta \tag{A.16}$$

$$\mu \quad free \tag{A.17}$$

Le *min-max regret* MCLP est NP-difficile car résoudre ce problème pour un seul scénario signifie résoudre un problème NP-difficile. Ainsi, le problème de décision de *min-max regret* MCLP est dans P si et seulement si $P = NP$.

Chapitre 5: Algorithmes

Ce chapitre est dédié à la description des algorithmes exacts, heuristiques et hybrides pour le *min-max regret* WSCP et *min-max regret* MCLP. Les algorithmes exacts proposés par [Pereira and Averbakh, 2013] pour le *min-max regret* WSCP ont été reproduits dans cette thèse et les détails sont fournis ci-après. Puis, les adaptations réalisées pour résoudre le *min-max regret* MCLP sont aussi présentées. Ensuite, les heuristiques proposées pour le *min-max regret* WSCP et le *min-max regret* MCLP sont décrites. Finalement, les méthodes hybrides couplant des algorithmes exacts à des heuristiques sont introduites.

Algorithmes exactes

L'algorithme exact pour le *min-max regret* WSCP de [Pereira and Averbakh, 2013] est un algorithme de plans de coupes inspiré de la décomposition de Benders [Benders, 1962], appelé BLD, de l'anglais *Benders-like Decomposition algorithm.* La méthode est similaire à celle appliquée pour résoudre le problème du voyageur de commerce avec critère d'optimisation min-max regret [Montemanni et al., 2007], le problème du sac-à-dos avec la fonction objectif min-max regret [Furini et al., 2015] et le problème du plus court chemins restreint avec une optimisation du type min-max regret [Assunção et al., 2016].

Benders-like decomposition pour le problème robuste WSCP

La méthode BLD pour le *min-max regret* WSCP repose sur la formulation mathématique (A.4)-(A.5) et (A.6)-(A.8). Le nombre de contraintes (A.7) est exponentiel. Elles sont relaxées et remplacées par (A.18) dans le problème maître de la façon suivante: soit le sous-ensemble de solutions $\Gamma^h \subseteq \Gamma$ induit par les contraintes (A.18). Á chaque itération, une nouvelle contrainte est séparée de $\Gamma \setminus \Gamma^h$ à partir de la résolution du sous-problème WSCP. Puis, elle est ajoutée au problème maître. BLD s'arrête lorsque la limite inférieure obtenue par le problème maître est égale à la limite supérieure, ou bien lorsque la limite de temps définie en amont est atteinte.

$$\theta \leq \sum_{j \in M} u_j y_j + \sum_{j \in M} y_j \left(l_j - u_j\right) x_j \quad \forall y \in \Gamma^h \tag{A.18}$$

Benders-like decomposition pour le problème robuste MCLP

La méthode BLD pour le *min-max regret* MCLP est fondée sur le modèle mathématique (A.10)-(A.12) et (A.15)-(A.17). Le nombre de contraintes (A.16) est exponentiel. De ce fait, elles sont relaxées et remplacées par (A.16) dans le problème maître comme suit. Soit $\Delta^h \subseteq \Delta$ l'ensemble de solutions induites par les contraintes (A.19). À chaque itération, une nouvelle contrainte est séparée dans $\Delta \setminus \Delta^h$, à partir de la résolution du sous-problème MCLP et ensuite intégrée au problème maître. BLD s'arrête lorsque la limite inférieure obtenue par le problème maître est égale à la limite supérieure, ou si la limite de temps initialement définie est atteinte.

$$\mu \geq \sum_{j \in M} u_j y_j + \sum_{j \in M} y_j \left(l_j - u_j\right) x_j \quad \forall y \in \Delta^h \tag{A.19}$$

Benders Étendu

Pereira and Averbakh [2013] a montré que la convergence de la méthode BLD peut être lente, car une seule coupe est introduite à chaque exécution du problème maître. Afin d'améliorer cela, Pereira and Averbakh [2013] ont développé un BLD étendu pour le *min-max regret* WSCP, noté ici comme EB, de l'anglais Extended Benders. La méthode EB est basée sur le travail de Fischetti et al. [2010] où les solutions courantes sont calculées par un solveur commercial de PLNE (CPLEX) et utilisées pour générer des nouvelles coupes, qui sont par la suite introduites dans le problème maître. Ainsi, le nombre espéré d'itérations du EB est plus petit que pour le BLD. Par conséquent, la convergence peut être accélérée.

Branch and Cut

B&C a été appliqué au *min-max regret* WSCP par Pereira and Averbakh [2013] et au *min-max regret* MCLP dans cette thèse. B&C pour le *min-max regret* WSCP repose sur la relaxation linéaire du modèle mathématique (A.4)-(A.5), (A.6), (A.8) et (A.18), tandis que la formulation (A.10)-(A.12), (A.15), (A.17) et (A.19) est utilisée dans le B&C pour le *min-max regret* MCLP. La méthode est similaire dans les deux cas. De ce fait, seul le B&C pour le *min-max regret* MCLP est décrit ci-après. L'algorithme démarre avec la formulation donnée par l'ensemble $\Delta' = \Delta^1$ de contraintes (A.19). Lorsqu'une solution entière (X', μ') est trouvée dans un noeud de l'arbre d'énumération, la nouvelle solution $Y^{s(X')}$ est calculée pour le sous-problème MCLP dans le scénario $s(X')$. Puis, $Y^{s(X')}$ est intégrée à Δ' et une nouvelle coupe est utilisée pour tous les noeuds actives dans l'arbre de branch-and-bound. Ainsi, l'algorithme B&B n'est pas réinitialisé pour $\Delta' \cup \{Y^{s(X')}\}$. L'algorithme est correct car pour chaque solution X' trouvée, une nouvelle contrainte (4.24) est générée pour assurer le bon calcul de la valeur μ' [Montemanni et al., 2007].

Heuristiques

Plusieurs heuristiques ont été développées: basées sur des scénarios spécifiques [Kasperski and Zieliński, 2006; Coco et al., 2015], path relinking [Glover and Laguna, 1993], pilot method [Voss et al., 2005] et basées sur la programmation linéaire [Dantzig, 1963]. Les solutions obtenues pour ces heuristiques, à l'exception de celle basée sur la programmation linéaire est 2-approximative.

Heuristiques basées en scénarios spécifiques

L'heuristique appelée AMU, de l'anglais *Algorithm Mean Upper* (AMU) a été proposée par Kasperski and Zieliński [2006]. On considère le scénario moyen s^m, donné par $b_j^m(c_j^m) = (l_j + u_j)/2$ et le scénario dans la limite supérieure s^u, défini par $b_j^u(c_j^u) = u_j$. La preuve que $\rho^{s(X^m)}(X^m) \leq 2 \times \rho^{s(X^*)}(X^*)$ peut être obtenue à partir de l'heuristique AMU de Kasperski and Zieliński [2006]. Les algorithmes pour le *min-max regret* WSCP (resp. *min-max regret* MCLP) consistent à résoudre une seule instance du WSCP (resp. MCLP) pour deux scénarios spécifiques, c-à-d le scénario moyen et le scénario dans la limite supérieure. Puis, la meilleure solution parmi X^m et X^u est retournée par l'algorithme. Il est important de souligner qu'une heuristique similaire a été introduite aussi par Kasperski and Zieliński [2006], appelé AM, de l'anglais *Algorithm Mean* (AM) qui est simplement une variation de AMU, où seul le scénario s^m est considéré.

L'heuristique basée sur des scénarios spécifiques, SBA de l'anglais Scenario Based Algorithm (SBA) a été introduite par Coco et al. [2015] et appliquée avec succès par Carvalho et al. [2016]; Coco et al. [2016]. SBA est une généralisation de AM, où un ensemble Q de scénarios est étudié, au lieu d'un scénario unique. L'algorithme consiste à résoudre une instance pour le problème ciblé (eg. MCLP) pour chaque scénario dans Q, et à retourner la meilleure solution trouvée, *i.e.* de regret maximal est minimal. On considère s_p le scénario où $b_j^{s_p} = \{l_j + (p \times (u_j - l_j))$ pour chaque $j \in M\}$. Alors, $Q = \{s_p \mid p = \frac{i}{q}$ et $i = 0, 1, 2, 3, \cdots, q\}$, avec un nombre q de pas SBA fixés à 100. Ainsi, le scénario moyen est toujours utilisé. De ce fait, les solutions obtenues pour le SBA sont au moins aussi bonnes que les solutions produites par AM.

Path relinking

Path Relinking (PR) [Glover and Laguna, 1993] est une heuristique de recherche qui a été appliquée avec des très bons résultats à de nombreux problèmes d'optimisation [Glover et al., 2000; Prins et al., 2006; Resende et al., 2010]. On considère deux solutions X^s et X^f. L'idée de PR est de transformer progressivement la solution X^s en X^f, à partir de l'application de petits mouvements, dans l'espoir que lors de la transformation, de nouvelles solutions de bonnes qualités soient découvertes. Ce mécanisme est motivé par le fait que des solutions quasi-optimales ont des composants similaires. Pour le *min-max regret* WSCP et le *min-max regret* MCLP, cela se traduit par des sous-ensembles de colonnes qui apparaissent de façon récurrente dans des optimum locaux, *i.e.* dans le *min-max regret* WSCP et le *min-max regret* MCLP deux solutions X^s and X^f peuvent avoir un sous ensemble de colonnes L tel que $L = X^s \cap X^f$ et $L \neq \emptyset$. Ainsi, PR utilise cette information pour créer une séquence de solutions intermédiaires entre X^s et X^f dans l'espoir de trouver des solutions de meilleure qualité.

Dans le PR pour le *min-max regret* WSCP et pour le *min-max regret* MCLP, un chemin de solutions entre X^i et X^f est créé en utilisant deux mouvements différents: (i) une colonne de X^i qui n'est pas dans X^f est supprimée de X^i. Puis des colonnes de X^f qui ne sont pas dans X^i sont ajoutées à X^i jusqu'à ce que les lignes de X^i soient couvertes par au moins une colonne (ii) une colonne de X^f qui n'appartient pas à la solution X^i est introduite en X^i, puis des colonnes redondantes de X^i sont supprimées.

PR a une mémoire qui stocke un ensemble de solutions réalisables. Cette mémoire est initialisée avec les 100 solutions calculées à partir de SBA. De plus, les nouvelles solutions trouvées pendant le PR sont aussi insérées dans la mémoire. Deux stratégies pour utiliser cette mémoire ont été développées. Dans la première stratégie (a) le PR est exécuté de la meilleure solution de la mémoire vers toutes les autres qui sont stockées. Dans la deuxième stratégie (b) le PR est exécuté pour toute paire de solutions de la mémoire. Les deux différents mouvements et les deux stratégies de gestion de la mémoire ont été combinés, ce qui a résulté en quatre variations du PR, appelées PR-R-Best (i-a), PR-I-Best (i-b), PR-R-Any (ii-a) et PR-I-Any (ii-b). R et I signifient respectivement, suppression (Remove) initiale d'une colonne et insertion (Insertion) initiale d'une colonne. *Best* et *All* indiquent respectivement si PR est appliqué en utilisant la stratégie (a) ou (b).

Pilot Method

La méthode Pilot Method (PM) [Duin and Voss, 1999; Voss et al., 2005] est une méta-heuristique qui utilise une heuristique constructive gloutonne H pour construire une heuristique plus performante H' de la façon suivante. Étant donné une heuristique constructive, la méthode insère un élément à la fois dans la solution partielle. Cependant, au lieu d'utiliser un critère d'évaluation glouton et local, le critère utilisé par H' fonctionne comme suit: (i) insérer l'élément individuellement dans la solution, (ii) appliquer H jusqu'à ce qu'une solution réalisable soit trouvée et (iii) utilise le coût de cette solution comme référence d'un coût pour insérer un élément. Á chaque itération, ces trois pas sont exécutés pour un élément candidat et la meilleure insertion (celle avec le meilleure coût) est réalisée. Cette méthode, proposée par Coco et al. [2014a] est utilisée ici pour résoudre le *min-max regret* WSCP et le *min-max regret* MCLP. Comme suggéré par Coco et al. [2014a], l'heuristique H utilisée est l'AM [Kasperski and Zieliński, 2006].

Algorithme basé sur la programmation linéaire

Assunção et al. [2017] a développé une heuristique basée sur la programmation linéaire [Dantzig, 1963] pour le problème du plus court chemin restreint avec critère d'optimisation *min-max regret*. Elle est référencée ici comme LPA, de l'anglais Linear Programming Algorithm et été adaptée et utilisée pour le *min-max regret min-max regret* WSCP et pour le *min-max regret* MCLP.

Algorithmes hybrides

SBA est utilisé comme point de démarrage pour les algorithmes exacts développés. SBA peut retourner au plus une centaine de solutions. Elles sont fournies dans la première itération au EB et aussi comme racine pour le démarrage de l'arbre de branch-and-bound du B&C, *i.e.*, les solutions calculée par SBA sont intégrées dans Γ^1 ou Δ^1. Ainsi, l'algorithme exact démarre dans la première itération avec un Γ^1 ou Δ^1 plus grand.

Dans l'EB, PR est appelé à chaque itération h, tandis que dans le B&C, le PR est appliqué seulement lorsqu'une solution réalisable est trouvée pendant l'exécution de l'arbre de branch-and-bound. Dans les deux cas, PR réalise une recherche pour identifier de nouvelles solutions à partir de celles déjà découvertes par cette méthode. Par conséquent, des solutions différentes sont ajoutées à Γ^h ou Δ^h à chaque itération. Les stratégies PR peuvent être intégrées très facilement dans des algorithmes exacts.

Chapitre 7: Résultats, conclusions et perspectives futures

Les problèmes de couverture robuste *min-max regret* WSCP [Pereira and Averbakh, 2013] et le *min-max regret* MCLP ont été étudiés dans cette thèse. Même si les deux problèmes ont pour objectif de minimiser le regret maximal, les méthodes développées (Chapitre 5) ont produit des résultats très différents. En particulier, des méthodes qui ont bien fonctionné pour le *min-max regret* n'ont pas forcément produit des résultats performants pour l'autre problème. Ceci est dû très probablement à la qualité de la relaxation linéaire du modèle pour le *min-max regret* MCLP et l'impact de la contrainte additionnelle pour limiter le nombre de colonnes (tentes médicales dans le cadre applicatif) à intégrer la solution.

Concernant le *min-max regret* WSCP, l'algorithme exact de Pereira and Averbakh [2013] a résolu 8 des 75 instances, tandis que l'algorithme hybride que nous avons proposé a résolu à l'optimalité 5 instances de plus. L'intégration d'heuristiques dans les algorithmes exacts n'a pas résulté en une amélioration significative en termes de qualité des solutions finales produites. Ceci étant, une piste pour des études futures consiste à utiliser les heuristiques pour réduire le nombre de contraintes 4.9.

Les cinq heuristiques proposées pour le *min-max regret* WSCP ont produit en moyenne de meilleures solutions que l'heuristique de Pereira and Averbakh [2013]. De plus, elles ont trouvé des limites supérieures plus intéressantes que les méthodes exactes pour la moitié des instances. Les deux heuristiques avec les meilleures performances sont celles basées sur la programmation linéaire et SBA, avec des résultats remarquables pour la première et des temps de calcul très faibles pour la deuxième. Une caractéristique importante est que ces heuristiques sont génériques et peuvent être appliquées à n'importe quel problème OR de type *min-max regret*

Le *min-max regret* MCLP a été introduit dans cette thèse et est une généralisation du MCLP. Le modèle proposé trouve des application en logistique de crise, où des tentes médicales doivent être localisées afin d'atténuer la surcharge de demande aux hôpitaux. Les incertitudes sont associées aux nombre de personnes cherchant de l'aide après une catastrophe. L'aspect additionnel au MCLP est que les coûts des colonnes sont considérés incertains et de plus une contrainte pour limiter le nombre de colonnes (tentes) a été aussi ajoutée.

Des scénarios réalistes pour le tremblement de terre qui a frappé Katmandou au Nepal en 2015 ont été utilisés dans les expérimentations. Les modèles proposés fournissent un éventail de possibilités pour obtenir des solutions de qualité et peuvent fournir de l'aide à la décision post-catastrophe. L'intérêt est d'obtenir une vision globale et des solutions robustes en considérant un nombre important de paramètres complexes tels que les incertitudes sur la demande et la limite dans le nombre de tentes, avant même d'appliquer la solution sur place. Ainsi, les modèles sont un outil pour aider dans la prise de décisions complexes dans ce contexte.

Les résultats numériques indiquent que les instances réalistes ont été résolues très efficacement. En particulier, les algorithmes exacts BLD, EB et le B&C ont fourni des solutions pour les 100 scénarios testés en quelques secondes. En termes de qualité des solutions, B&C s'est montré plus intéressant que le BLD et l'EB. De plus, le B&C a été plus rapide et aussi plus simple à mettre en œuvre. Le résultat des heuristiques est aussi intéressant, mais vu la performance des méthodes exacts pour les instances réalistes, l'utilisation des heuristiques peut être réservée au cas où des instances de plus grande taille, ou plus difficiles, apparaissent dans le contexte de l'application.

En termes d'application réelle, plusieurs directions de recherche ont émergé. Notamment, l'utilisation de la simulation pour étudier le phénomène de files d'attente dans les tentes médicales et estimer le temps nécessaire pour apporter des soins à la population. Il y a aussi des opportunités en termes de modèles d'optimisation combinatoire. Notamment, l'intégration d'autres contraintes pour la gestion des ressources humaines, financières et matérielles. Un problème de gestion de micro- économie nous a aussi été signalé et consiste à définir le temps que les tentes médicales doivent être disponibles à la population après une catastrophe. En effet, dans certains pays, la présence de tentes médicales avec des soins gratuits à fortement perturbé les hôpitaux privés installés, avec des conséquences importantes pour l'économie locale.

Les perspectives de développement pour les méthodes sont aussi riches. Par exemple, il y a un boulevard d'opportunités ouvert pour le développement de nouvelles coupes et inégalités valides pour le *min-max regret* WSCP et pour le *min-max regret* MCLP. Entre autres, il est aussi intéressant d'étudier s'il est possible de produire des PLNE compacts pour les problèmes d'OR avec critère d'optimisation *min-max regret*, ainsi que de déployer des heuristiques proposées à d'autres problèmes d'OR en utilisant le critère d'optimisation *min-max regret*.

Appendix B

Full Tables

Instance	BLD				EB				B&C			
	$\rho(X)$	Gap	T (s)	Cuts	$\rho(X)$	Gap	T (s)	Cuts	$\rho(X)$	Gap	T (s)	Cuts
BKZ 41	13526	24.49	900.00	21	13448	17.54	900.00	124	12910	14.88	900.00	245
BKZ 42	13115	14.66	900.00	22	13115	9.41	900.00	132	13115	10.11	900.00	587
BKZ 43	15927	25.80	900.00	19	15927	18.56	900.00	134	15452	17.79	900.00	577
BKZ 44	14769	23.69	900.00	17	14769	16.23	900.00	104	14474	16.15	900.00	432
BKZ 45	14109	17.03	900.00	23	14109	9.69	900.00	158	13862	9.83	900.00	223
BKZ 46	13050	20.96	900.00	26	13050	14.27	900.00	140	12773	13.26	900.00	68
BKZ 47	13615	19.65	900.00	32	13500	13.59	900.00	179	13308	12.96	900.00	53
BKZ 48	12541	19.60	900.00	28	12541	13.53	900.00	172	12167	13.89	900.00	151
BKZ 49	12996	14.49	900.00	25	12996	7.78	900.00	170	12833	9.07	900.00	119
BKZ 410	14909	27.25	900.00	21	14909	20.60	900.00	132	14279	19.42	900.00	78
BKZ 51	10755	11.08	900.00	23	10755	4.94	900.00	131	10725	7.10	900.00	1004
BKZ 52	10617	10.38	900.00	20	10617	6.37	900.00	128	10451	6.32	900.00	776
BKZ 53	11070	16.91	900.00	20	11070	10.92	900.00	125	10927	8.92	900.00	536
BKZ 54	11153	15.43	900.00	18	11153	8.42	900.00	142	11153	10.49	900.00	1308
BKZ 55	12048	16.43	900.00	14	12048	9.52	900.00	82	12020	10.70	900.00	1279
BKZ 56	10590	8.42	900.00	25	10590	3.72	900.00	163	10561	3.12	900.00	586
BKZ 57	11982	16.14	900.00	19	11982	10.56	900.00	126	11982	11.43	900.00	832
BKZ 58	12232	14.21	900.00	14	12232	8.91	900.00	101	12232	11.35	900.00	813
BKZ 59	11444	11.41	900.00	21	11444	6.19	900.00	137	11437	6.40	900.00	813
BKZ 510	11291	11.71	900.00	17	11291	5.43	900.00	120	11291	8.56	900.00	784
BKZ 61	5995	4.94	900.00	21	**5995**	**0.00**	**660.23**	114	**5995**	**0.00**	**211.66**	364
BKZ 62	6255	4.65	900.00	18	**6235**	**0.00**	**618.34**	77	**6235**	**0.00**	**220.14**	242
BKZ 63	6953	12.18	900.00	13	6953	5.41	900.00	103	6895	3.82	900.00	372
BKZ 64	7020	13.28	900.00	15	7020	5.63	900.00	95	**6895**	**0.00**	**876.78**	234
BKZ 65	6378	7.03	900.00	18	**6250**	**0.00**	**719.35**	114	**6250**	**0.00**	**214.47**	372
BKZ 41 b	16942	25.56	900.00	14	16942	19.09	900.00	94	16573	21.58	900.00	348
BKZ 42 b	16136	18.80	900.00	16	16136	13.06	900.00	97	16136	16.26	900.00	290
BKZ 43 b	17201	23.85	900.00	14	17201	19.23	900.00	77	17004	20.66	900.00	339
BKZ 44 b	17987	31.10	900.00	16	17987	23.24	900.00	93	17654	25.34	900.00	384
BKZ 45 b	18590	32.12	900.00	12	18590	25.96	900.00	76	18286	28.55	900.00	423
BKZ 46 b	15753	21.54	900.00	14	15753	18.12	900.00	99	15590	17.84	900.00	194
BKZ 47 b	16546	22.62	900.00	15	16546	16.03	900.00	95	16546	17.78	900.00	57
BKZ 48 b	15197	15.68	900.00	24	15197	11.23	900.00	126	15197	12.29	900.00	171
BKZ 49 b	17337	26.76	900.00	13	17337	22.79	900.00	88	17064	21.71	900.00	98
BKZ 410 b	17250	26.15	900.00	14	17250	20.36	900.00	107	16926	19.46	900.00	65
BKZ 51 b	13321	16.50	900.00	14	13321	11.69	900.00	103	13321	13.60	900.00	1065
BKZ 52 b	11687	14.09	900.00	16	11687	8.61	900.00	120	11488	7.33	900.00	807
BKZ 53 b	11972	14.19	900.00	15	11972	7.67	900.00	85	11972	10.30	900.00	619
BKZ 54 b	12423	10.57	900.00	20	12423	6.30	900.00	112	12403	5.40	900.00	657
BKZ 55 b	12616	13.68	900.00	14	12616	6.24	900.00	106	12572	8.39	900.00	1426
BKZ 56 b	12940	13.46	900.00	14	12940	10.73	900.00	79	12940	10.32	900.00	622
BKZ 57 b	12428	12.31	900.00	15	12428	7.63	900.00	79	12424	8.24	900.00	597
BKZ 58 b	12622	16.73	900.00	13	12622	12.91	900.00	78	12494	13.22	900.00	584
BKZ 59 b	12576	15.57	900.00	16	12576	8.01	900.00	86	12576	9.77	900.00	694
BKZ 510 b	12011	17.17	900.00	13	12011	10.86	900.00	96	12011	14.33	900.00	1036
BKZ 61 b	7600	19.84	900.00	12	7600	12.44	900.00	92	7571	14.76	900.00	353
BKZ 62 b	6615	10.73	900.00	11	6499	2.03	900.00	95	**6499**	**0.00**	**760.29**	547
BKZ 63 b	7030	11.91	900.00	13	7030	7.46	900.00	86	7021	4.40	900.00	414
BKZ 64 b	7339	12.87	900.00	12	7339	5.78	900.00	66	7299	6.03	900.00	157
BKZ 65 b	6940	4.44	900.00	22	**6892**	**0.00**	**899.32**	105	**6892**	**0.00**	**308.02**	312
BKZ 41 c	13617	19.76	900.00	24	13617	14.46	900.00	153	13617	17.39	900.00	1065
BKZ 42 c	13926	23.97	900.00	17	13926	16.70	900.00	106	13637	17.65	900.00	807
BKZ 43 c	14444	20.21	900.00	17	14444	14.38	900.00	93	14110	15.06	900.00	619
BKZ 44 c	14774	21.00	900.00	16	14774	14.50	900.00	113	14774	16.08	900.00	657
BKZ 45 c	15555	26.68	900.00	19	15555	18.51	900.00	109	15314	20.12	900.00	1426
BKZ 46 c	13334	19.87	900.00	17	13334	12.25	900.00	105	13334	14.61	900.00	622
BKZ 47 c	13585	20.97	900.00	24	13423	13.29	900.00	140	13170	13.24	900.00	597
BKZ 48 c	15406	20.33	900.00	16	15406	14.13	900.00	97	15337	15.38	900.00	584
BKZ 49 c	14809	22.75	900.00	17	14809	16.57	900.00	117	14523	15.8	900.00	694
BKZ 410 c	14459	21.59	900.00	22	14459	16.01	900.00	123	14347	18.03	900.00	1036
BKZ 51 c	10525	14.09	900.00	17	10525	7.69	900.00	118	10525	8.64	900.00	353
BKZ 52 c	11753	16.71	900.00	17	11753	10.26	900.00	122	11581	12.22	900.00	547
BKZ 53 c	11139	15.20	900.00	18	11139	9.75	900.00	129	10844	7.11	900.00	414
BKZ 54 c	9376	7.35	900.00	26	9187	0.24	900.00	205	**9187**	**0.00**	**183.67**	157
BKZ 55 c	11088	12.09	900.00	20	11088	6.94	900.00	141	10965	6.40	900.00	312
BKZ 56 c	12350	21.58	900.00	14	12350	13.86	900.00	78	12311	14.53	900.00	658
BKZ 57 c	11590	13.52	900.00	16	11590	7.94	900.00	104	11590	9.99	900.00	329
BKZ 58 c	11738	17.37	900.00	20	11738	11.02	900.00	146	11529	9.21	900.00	458
BKZ 59 c	11949	19.67	900.00	16	11949	13.65	900.00	114	11791	13.60	900.00	447
BKZ 510 c	11724	15.56	900.00	17	11724	11.56	900.00	118	11724	12.50	900.00	618
BKZ 61 c	6822	10.30	900.00	16	6800	2.86	900.00	119	6800	3.86	900.00	103
BKZ 62 c	6450	9.43	900.00	15	6366	0.62	900.00	132	**6366**	**0.00**	**433.88**	95
BKZ 63 c	7028	11.45	900.00	16	6940	5.07	900.00	106	6940	3.88	900.00	149
BKZ 64 c	7062	13.45	900.00	14	7031	7.05	900.00	103	6989	5.81	900.00	213
BKZ 65 c	6757	10.07	900.00	15	6750	3.20	900.00	119	6746	2.29	900.00	140
Average		16.73				10.63				11.02		

Table B.1: Comparison among the exact algorithms to the set BKZ.

Instance	$\rho(X^{B\&C})$	PR R Best		PR R Any		PR I Best		PR I Any	
		Dev (%)	T (s)	Dev (%)	T(s)	Dev (%)	T (s)	Dev (%)	T (s)
BKZ 41	12910	2.16	49.59	**1.98**	264.05	2.20	54.09	2.20	261.94
BKZ 42	13115	0.00	8.61	0.00	30.64	0.00	8.56	0.00	37.00
BKZ 43	15452	1.50	31.64	1.50	103.54	1.50	31.74	1.50	107.73
BKZ 44	14474	1.05	51.53	1.05	150.55	1.05	50.19	1.05	141.17
BKZ 45	13862	**0.87**	14.58	**0.87**	51.33	1.78	19.90	1.46	65.35
BKZ 46	12773	0.39	12.69	0.39	44.34	0.39	13.38	0.39	42.74
BKZ 47	13308	0.37	18.16	0.37	74.78	0.37	15.38	0.37	66.13
BKZ 48	12167	1.92	8.65	1.92	15.21	1.92	8.78	1.92	15.18
BKZ 49	12833	1.17	15.31	**0.35**	83.85	1.17	15.87	1.17	64.60
BKZ 410	14279	0.20	16.55	0.20	41.39	0.20	16.68	0.20	40.11
BKZ 51	10725	0.00	18.50	0.00	85.50	0.00	17.99	0.00	76.24
BKZ 52	10451	0.00	10.71	0.00	34.67	0.00	10.28	0.00	34.76
BKZ 53	10927	**1.21**	12.40	**1.21**	43.56	1.31	12.25	1.31	40.11
BKZ 54	11153	0.00	17.33	0.00	68.64	0.00	18.21	0.00	67.66
BKZ 55	12020	0.32	19.38	0.32	60.09	0.32	19.82	0.32	60.23
BKZ 56	10561	0.27	11.99	0.27	62.46	0.27	12.25	0.27	64.97
BKZ 57	11982	0.00	16.23	0.00	42.00	0.00	16.05	0.00	42.25
BKZ 58	12232	0.00	23.30	0.00	67.23	0.00	23.57	0.00	66.02
BKZ 59	11437	**0.00**	17.87	**0.00**	94.86	0.06	15.86	0.06	74.28
BKZ 510	11291	0.00	15.10	0.00	33.84	0.00	14.01	0.00	33.91
BKZ 61*	5995	0.00	39.29	0.00	170.79	0.00	36.70	0.00	168.57
BKZ 62*	6235	0.32	71.77	0.32	208.06	0.32	78.22	0.32	217.18
BKZ 63	6895	0.52	58.96	0.52	107.55	0.52	59.45	0.52	115.05
BKZ 64*	6895	2.71	67.08	**2.41**	205.77	2.71	73.96	**2.41**	204.08
BKZ 65*	6250	1.82	65.86	**0.82**	240.55	1.82	68.38	1.82	195.12
BKZ 41 b	16573	**-0.63**	28.28	**-0.63**	72.41	0.77	26.95	0.77	65.80
BKZ 42 b	16136	0.00	30.25	0.00	77.21	0.00	32.82	0.00	87.62
BKZ 43 b	17004	1.16	28.64	1.16	53.71	1.16	28.87	1.16	50.91
BKZ 44 b	17654	0.13	43.30	0.13	113.91	0.13	42.89	0.13	111.66
BKZ 45 b	18286	0.96	42.79	0.96	99.63	0.96	39.90	0.96	90.99
BKZ 46 b	15590	0.00	23.94	0.00	44.34	0.00	23.55	0.00	43.01
BKZ 47 b	16546	0.00	25.53	0.00	42.39	0.00	24.63	0.00	40.56
BKZ 48 b	15197	0.00	12.56	0.00	22.39	0.00	12.86	0.00	22.20
BKZ 49 b	17064	**-0.94**	24.45	**-0.94**	53.97	0.07	22.90	0.07	42.30
BKZ 410 b	16926	0.03	24.84	0.03	63.41	0.03	24.97	0.03	63.49
BKZ 51 b	13321	0.46	16.29	0.46	48.82	0.46	16.26	0.46	47.92
BKZ 52 b	11488	1.03	10.18	1.03	35.99	1.03	10.23	1.03	35.69
BKZ 53 b	11972	0.04	20.31	0.04	52.71	0.04	21.22	**-0.17**	65.14
BKZ 54 b	12403	0.16	28.38	0.16	113.72	0.16	29.19	0.16	114.34
BKZ 55 b	12572	0.53	14.71	0.53	44.98	0.53	14.86	0.53	42.73
BKZ 56 b	12940	0.12	16.12	0.12	48.88	0.12	16.54	0.12	47.90
BKZ 57 b	12424	0.03	11.72	0.03	64.86	0.03	11.75	0.03	63.70
BKZ 58 b	12494	0.37	51.58	0.37	141.85	0.37	51.80	0.37	139.89
BKZ 59 b	12576	0.00	14.19	0.00	63.63	0.00	14.74	0.00	62.93
BKZ 510 b	12011	0.00	24.93	0.00	61.06	0.00	24.80	0.00	58.54
BKZ 61 b	7571	0.52	62.69	0.52	93.02	0.52	63.07	0.52	92.88
BKZ 62 b	6499	1.78	60.80	1.78	96.32	1.78	61.94	1.78	95.69
BKZ 63 b*	7021	0.13	68.25	0.13	166.59	0.13	71.94	0.13	183.68
BKZ 64 b	7299	0.55	61.92	0.55	108.40	0.55	61.53	0.55	103.09
BKZ 65 b*	6892	0.70	26.24	0.70	48.88	0.70	26.08	0.70	47.53
BKZ 41 c	13617	0.00	10.08	0.00	31.58	0.00	9.90	0.00	30.86
BKZ 42 c	13637	0.13	75.77	0.13	282.84	0.13	74.00	0.13	277.84
BKZ 43 c	14110	0.42	8.57	0.42	43.31	0.42	8.16	0.42	46.30
BKZ 44 c	14774	0.00	5.21	0.00	9.55	0.00	5.00	0.00	9.86
BKZ 45 c	15314	0.59	15.32	0.59	76.50	0.59	17.09	0.59	76.05
BKZ 46 c	13334	0.00	19.41	0.00	40.46	0.00	19.60	0.00	40.40
BKZ 47 c	13170	0.41	10.16	0.41	38.59	0.41	9.64	0.41	38.48
BKZ 48 c	15337	1.90	15.75	1.90	48.08	1.90	13.18	1.90	42.84
BKZ 49 c	14523	0.83	19.14	0.83	60.24	0.83	18.69	0.83	59.21
BKZ 410 c	14347	0.22	34.82	**-0.52**	114.29	0.78	27.75	**-0.52**	111.30
BKZ 51 c	10525	0.00	11.22	0.00	36.21	0.00	11.37	0.00	35.46
BKZ 52 c	11581	0.62	40.13	**0.16**	154.62	0.62	38.47	0.62	134.23
BKZ 53 c	10844	**0.26**	19.20	**0.26**	117.90	0.53	19.37	0.53	104.75
BKZ 54 c*	9187	1.63	6.83	1.63	23.93	1.63	6.27	1.63	23.41
BKZ 55 c	10965	0.00	28.73	0.00	96.08	0.00	31.94	0.00	102.94
BKZ 56 c	12311	1.19	69.89	1.19	225.93	1.19	81.15	1.19	242.53
BKZ 57 c	11590	0.00	20.52	0.00	66.81	0.00	19.54	0.00	58.67
BKZ 58 c	11529	0.62	14.48	0.62	56.57	0.62	14.70	0.62	57.13
BKZ 59 c	11791	0.28	36.81	0.28	142.18	0.28	36.90	0.28	136.50
BKZ 510 c	11724	0.00	15.76	0.00	45.57	0.00	15.68	0.00	45.59
BKZ 61 c	6800	0.41	108.02	0.41	486.83	0.41	108.20	0.41	472.35
BKZ 62 c*	6366	0.00	76.83	0.00	223.82	0.00	83.16	0.00	241.49
BKZ 63 c	6940	0.97	73.30	0.97	257.40	0.97	76.07	0.97	240.52
BKZ 64 c	6989	0.19	88.80	0.19	317.67	0.19	87.29	0.19	298.73
BKZ 65 c	6746	0.00	79.66	0.00	208.18	0.00	76.81	0.00	210.26
Average		0.29	31.61	**0.25**	98.98	0.35	31.97	0.32	96.80

Table B.2: Comparison among the Path Relinking strategies to the set BKZ.

Instance	$\rho(X^{B\&C})$	AMU		SBA		PR R Any		LPH		PM	
		Dev (%)	T (s)	Dev (%)	T(s)	Dev (%)	T (s)	Dev (%)	T (s)	Dev (%)	T (s)
BKZ 41	12910	4.77	2.59	2.20	17.20	1.98	264.05	**0.22**	170.1	0.75	148.26
BKZ 42	13115	0.00	1.80	0.00	7.46	0.00	30.64	0.00	23.78	0.00	58.11
BKZ 43	15452	3.07	3.28	2.02	20.64	1.50	103.54	**-0.27**	230.13	1.24	230.74
BKZ 44	14474	2.04	4.66	1.05	24.98	1.05	150.55	**0.00**	248.87	1.31	143.28
BKZ 45	13862	1.78	2.12	1.78	9.00	0.87	51.33	**-0.61**	26.42	1.60	96.21
BKZ 46	12773	2.17	2.18	0.39	9.48	0.39	44.34	**0.09**	73.05	**0.09**	72.77
BKZ 47	13308	2.31	1.92	0.37	7.56	0.37	74.78	**0.02**	74.00	0.10	60.97
BKZ 48	12167	3.07	1.88	1.92	7.53	1.92	15.21	**0.00**	51.94	2.38	76.00
BKZ 49	12833	1.27	1.48	1.27	8.15	0.35	83.85	**-0.95**	15.13	0.71	101.46
BKZ 410	14279	4.41	2.86	0.20	13.86	0.20	41.39	**-0.61**	271.36	0.00	82.69
BKZ 51	10725	0.28	1.99	**0.00**	10.74	**0.00**	85.50	**0.00**	21.63	**0.00**	145.07
BKZ 52	10451	1.59	2.24	**0.00**	8.10	**0.00**	34.67	**0.00**	16.94	0.26	90.33
BKZ 53	10927	1.31	1.69	1.31	7.73	1.21	43.56	**-0.10**	37.35	0.33	136.99
BKZ 54	11153	**0.00**	2.70	**0.00**	10.12	**0.00**	68.64	**0.00**	51.57	2.70	236.78
BKZ 55	12020	0.23	2.83	**-0.32**	14.57	**-0.32**	60.09	**-0.32**	104.98	**-0.32**	184.86
BKZ 56	10561	0.27	1.28	0.27	6.80	0.27	62.46	**0.00**	15.18	0.19	87.71
BKZ 57	11982	0.00	2.23	0.00	9.42	0.00	42.00	**-0.27**	62.41	0.47	143.65
BKZ 58	12232	0.00	2.95	0.00	17.39	0.00	67.23	0.00	97.24	**-0.08**	272.97
BKZ 59	11437	0.06	2.40	0.06	10.00	**0.00**	94.86	**0.00**	17.95	0.16	175.41
BKZ 510	11291	**0.00**	1.79	**0.00**	9.56	**0.00**	33.84	**0.00**	22.93	1.20	100.90
BKZ 61*	5995	0.00	5.37	0.00	26.86	0.00	170.79	0.00	29.73	0.00	453.75
BKZ 62*	6235	**0.32**	13.93	**0.32**	54.38	**0.32**	208.06	0.56	79.06	**0.32**	591.48
BKZ 63	6895	0.84	15.46	0.52	55.52	0.52	107.55	0.09	292.96	**0.00**	770.86
BKZ 64*	6895	1.81	12.47	2.71	50.52	2.41	205.77	**0.04**	137.74	4.26	851.17
BKZ 65*	6250	2.05	12.11	1.82	57.67	0.82	240.55	**0.00**	78.94	1.02	502.51
BKZ 41 b	16573	2.23	6.39	0.77	22.63	**-0.63**	72.41	**-0.63**	321.96	0.29	157.76
BKZ 42 b	16136	**0.00**	3.19	**0.00**	20.76	**0.00**	77.21	**0.00**	183.92	0.32	162.43
BKZ 43 b	17004	1.16	4.45	1.16	26.07	1.16	53.71	**-0.01**	246.07	1.26	192.03
BKZ 44 b	17654	1.89	3.18	0.13	37.18	0.13	113.91	**-0.53**	496.32	**-0.53**	322.76
BKZ 45 b	18286	1.66	7.75	0.96	35.09	0.96	99.63	**-1.90**	479.12	0.77	226.90
BKZ 46 b	15590	1.05	3.69	0.00	20.69	0.00	44.34	**-0.32**	159.55	0.00	160.59
BKZ 47 b	16546	0.00	5.95	0.00	22.90	0.00	42.39	**-0.41**	293.08	**-0.41**	230.83
BKZ 48 b	15197	0.00	1.85	0.00	11.05	0.00	22.39	**-0.76**	47.94	0.01	97.77
BKZ 49 b	17064	1.60	2.55	0.36	20.49	**-0.94**	53.97	**-0.94**	239.29	0.19	154.31
BKZ 410 b	16926	1.91	2.84	**0.03**	20.30	**0.03**	63.41	**0.03**	235.63	**0.03**	126.06
BKZ 51 b	13321	0.00	2.87	**-0.46**	12.31	**-0.46**	48.82	**-0.46**	112.24	0.08	165.72
BKZ 52 b	11488	1.73	2.22	1.03	7.38	1.03	35.99	**-0.03**	21.81	0.00	107.92
BKZ 53 b	11972	0.00	2.77	0.04	15.07	0.04	52.71	**-0.17**	93.86	0.04	282.68
BKZ 54 b	12403	0.16	2.83	0.16	17.34	0.16	113.72	**0.00**	35.34	1.36	279.07
BKZ 55 b	12572	0.35	2.29	**-0.53**	11.54	**-0.53**	44.98	**-0.53**	27.17	**-0.53**	171.58
BKZ 56 b	12940	0.00	3.59	0.12	13.65	0.12	48.88	**-0.44**	21.51	0.12	115.56
BKZ 57 b	12424	0.03	0.91	0.03	6.42	0.03	64.86	**0.00**	18.81	0.03	67.46
BKZ 58 b	12494	1.02	5.30	0.37	41.63	0.37	141.85	**-0.03**	260.28	0.40	442.15
BKZ 59 b	12576	0.00	1.58	0.00	8.89	0.00	63.63	0.00	28.53	0.00	147.51
BKZ 510 b	12011	0.00	4.13	0.00	21.34	0.00	61.06	**-0.47**	120.05	0.00	281.50
BKZ 61 b	7571	0.38	31.13	0.52	55.71	0.52	93.02	**-0.40**	854.54	0.15	900.00
BKZ 62 b	6499	1.78	15.86	1.78	55.88	1.78	96.32	**0.00**	153.10	**0.00**	900.00
BKZ 63 b*	7021	**0.13**	17.30	**0.13**	57.55	**0.13**	166.59	**0.13**	249.19	2.71	673.19
BKZ 64 b	7299	0.55	24.56	0.55	56.13	0.55	108.40	**0.00**	627.43	0.47	900.00
BKZ 65 b*	6892	0.70	8.16	0.70	24.51	0.70	48.88	**0.00**	33.28	0.70	339.65
BKZ 41 c	13617	0.00	1.74	0.00	6.29	0.00	31.58	**-0.30**	48.96	0.59	60.27
BKZ 42 c	13637	2.12	5.62	0.13	35.98	0.13	282.84	**-1.09**	317.35	0.01	280.87
BKZ 43 c	14110	2.37	1.59	0.42	5.66	0.42	43.31	**-0.26**	82.05	0.00	43.72
BKZ 44 c	14774	0.00	1.00	0.00	4.50	0.00	9.55	**-0.23**	41.10	0.12	38.79
BKZ 45 c	15314	1.57	1.75	0.29	8.92	0.59	76.50	**-0.69**	227.82	0.34	123.21
BKZ 46 c	13334	0.00	3.96	0.00	16.90	0.00	40.46	**-0.44**	221.25	0.12	176.62
BKZ 47 c	13170	3.15	1.31	0.41	6.65	0.41	38.59	**-0.10**	43.66	0.12	39.24
BKZ 48 c	15337	0.45	1.90	**-1.90**	8.46	**-1.90**	48.08	**-1.90**	27.26	0.29	71.48
BKZ 49 c	14523	1.97	1.98	0.83	13.46	0.83	60.24	**-0.78**	136.5	0.00	149.51
BKZ 410 c	14347	0.78	3.49	0.78	19.80	0.52	114.29	**-0.45**	228.62	0.94	101.72
BKZ 51 c	10525	**0.00**	1.38	**0.00**	6.23	**0.00**	36.21	**0.00**	43.79	0.82	155.92
BKZ 52 c	11581	1.49	3.37	0.62	22.80	0.16	154.62	**-0.64**	63.49	0.58	375.31
BKZ 53 c	10844	2.72	2.42	0.53	11.07	0.26	117.90	**0.00**	48.13	0.53	166.09
BKZ 54 c*	9187	2.06	1.09	1.63	3.95	1.63	23.93	**0.00**	10.50	1.28	67.85
BKZ 55 c	10965	1.12	3.42	0.00	19.31	0.00	96.08	**-0.14**	53.72	0.00	386.14
BKZ 56 c	12311	0.32	8.43	**-1.19**	33.40	**-1.19**	225.93	**-1.19**	817.32	**-1.19**	638.73
BKZ 57 c	11590	0.00	2.98	0.00	14.45	0.00	66.81	0.00	34.16	0.00	169.26
BKZ 58 c	11529	1.81	1.75	0.62	9.66	0.62	56.57	**0.10**	98.42	0.47	161.56
BKZ 59 c	11791	1.34	3.83	0.28	25.94	0.28	142.18	0.18	145.56	**-0.65**	463.21
BKZ 510 c	11724	0.00	2.13	0.00	10.77	0.00	45.57	**-0.26**	87.31	**-0.26**	187.39
BKZ 61 c	6800	0.32	19.83	0.41	56.19	0.41	486.83	**0.00**	253.75	**0.00**	812.46
BKZ 62 c*	6366	1.32	17.46	**0.00**	58.30	**0.00**	223.82	**0.00**	127.14	**0.00**	831.12
BKZ 63 c	6940	1.27	13.89	0.97	57.85	0.97	257.40	**0.00**	153.12	0.73	712.71
BKZ 64 c	6989	3.06	24.29	0.19	57.76	0.19	317.67	0.19	656.51	**-0.54**	900.00
BKZ 65 c	6746	0.16	10.14	**0.00**	53.84	**0.00**	208.18	0.16	178.14	**0.00**	653.45
Average		1.09	5.52	0.37	22.48	0.25	98.98	**-0.24**	156.76	0.37	302.07

Table B.3: Comparison among the proposed heuristics to the set BKZ.

Instance	SBA+EB			SBA+B&C			PR+EB			PR+B&C		
	$\rho(X)$	Gap	T (s)	$\rho(X)$	Gap	T (s)	$\rho(X)$	Gap	T (s)	$\rho(X)$	Gap	T (s)
BKZ 41	13194	14.03	900.00	12885	12.48	900.00	13368	14.57	900.00	12938	14.02	900.00
BKZ 42	13115	8.41	900.00	13115	9.92	900.00	13115	7.53	900.00	13115	8.56	900.00
BKZ 43	15764	17.00	900.00	15764	21.70	900.00	15927	17.30	900.00	15411	16.77	900.00
BKZ 44	14626	13.06	900.00	14609	15.99	900.00	14769	13.31	900.00	14487	14.30	900.00
BKZ 45	14109	9.85	900.00	13777	7.74	900.00	14109	9.63	900.00	13862	8.51	900.00
BKZ 46	12823	10.99	900.00	12764	13.93	900.00	13050	13.25	900.00	12807	13.18	900.00
BKZ 47	13357	10.18	900.00	13298	11.90	900.00	13531	11.87	900.00	13308	14.17	900.00
BKZ 48	12401	9.85	900.00	12220	9.30	900.00	12541	13.17	900.00	12167	11.54	900.00
BKZ 49	12996	7.82	900.00	12746	5.58	900.00	12996	7.59	900.00	12746	7.77	900.00
BKZ 410	14307	15.29	900.00	14251	16.96	900.00	14833	18.95	900.00	14318	18.90	900.00
BKZ 51	10725	3.89	900.00	10725	3.28	900.00	10755	3.66	900.00	10725	4.08	900.00
BKZ 52	10451	3.49	900.00	10451	3.20	900.00	10617	4.86	900.00	10451	2.68	900.00
BKZ 53	11070	9.12	900.00	10927	7.26	900.00	10946	7.30	900.00	10927	6.76	900.00
BKZ 54	11153	7.50	900.00	11153	10.08	900.00	11153	7.72	900.00	11153	9.02	900.00
BKZ 55	11982	6.81	900.00	11982	9.29	900.00	12002	6.77	900.00	12020	9.22	900.00
BKZ 56	10590	2.67	900.00	10561	1.59	900.00	10590	2.02	900.00	10561	0.65	900.00
BKZ 57	11982	9.51	900.00	11982	10.71	900.00	11982	9.06	900.00	11982	11.85	900.00
BKZ 58	12232	7.56	900.00	12222	9.45	900.00	12232	6.24	900.00	12222	9.00	900.00
BKZ 59	11444	4.85	900.00	11437	4.55	900.00	11444	4.00	900.00	11437	5.39	900.00
BKZ 510	11291	4.44	900.00	11291	6.71	900.00	11291	3.74	900.00	11291	6.22	900.00
BKZ 61	**5995**	**0.00**	**582.56**	**5995**	**0.00**	**201.23**	**5995**	**0.00**	**370.31**	**5995**	**0.00**	**143.04**
BKZ 62	**6235**	**0.00**	**431.03**	**6235**	**0.00**	**140.99**	**6235**	**0.00**	**331.56**	**6235**	**0.00**	**114.84**
BKZ 63	6901	3.62	900.00	**6895**	**0.00**	**809.56**	6895	3.56	900.00	6895	1.82	900.00
BKZ 64	6928	4.16	900.00	**6895**	**0.00**	**607.59**	6895	1.76	900.00	7020	2.77	900.00
BKZ 65	**6250**	**0.00**	**626.63**	**6250**	**0.00**	**162.91**	**6250**	**0.00**	**552.85**	**6250**	**0.00**	**129.95**
BKZ 41 b	16893	18.51	900.00	16666	20.31	900.00	16942	18.91	900.00	16583	20.45	900.00
BKZ 42 b	16136	12.38	900.00	16136	14.99	900.00	16136	12.60	900.00	16136	14.07	900.00
BKZ 43 b	17201	19.41	900.00	17057	18.85	900.00	17201	16.74	900.00	17004	17.48	900.00
BKZ 44 b	17907	24.18	900.00	17768	26.25	900.00	17987	24.22	900.00	17644	23.60	900.00
BKZ 45 b	18111	22.12	900.00	18108	24.20	900.00	18590	23.52	900.00	17938	23.94	900.00
BKZ 46 b	15590	14.25	900.00	15540	15.81	900.00	15753	12.78	900.00	15559	15.02	900.00
BKZ 47 b	16546	15.68	900.00	16546	16.63	900.00	16546	13.65	900.00	16478	16.77	900.00
BKZ 48 b	15197	9.22	900.00	15197	10.24	900.00	15197	8.67	900.00	15162	11.47	900.00
BKZ 49 b	17125	17.76	900.00	17125	19.79	900.00	17337	19.52	900.00	16991	18.27	900.00
BKZ 410 b	16931	16.64	900.00	16896	18.09	900.00	17250	16.69	900.00	16896	20.50	900.00
BKZ 51 b	13260	11.31	900.00	13260	11.66	900.00	13321	10.76	900.00	13310	11.19	900.00
BKZ 52 b	11606	6.93	900.00	11488	5.48	900.00	11687	5.87	900.00	11485	6.58	900.00
BKZ 53 b	11967	6.50	900.00	11952	9.67	900.00	11972	6.21	900.00	11952	8.14	900.00
BKZ 54 b	12423	4.19	900.00	12403	4.33	900.00	12423	4.00	900.00	12406	4.90	900.00
BKZ 55 b	12505	6.07	900.00	12505	6.74	900.00	12616	5.78	900.00	12505	5.61	900.00
BKZ 56 b	12924	8.62	900.00	12883	8.10	900.00	12940	7.50	900.00	12883	8.91	900.00
BKZ 57 b	12428	6.09	900.00	12424	7.86	900.00	12428	5.64	900.00	12424	7.47	900.00
BKZ 58 b	12540	11.34	900.00	12540	12.18	900.00	12622	10.44	900.00	12544	10.54	900.00
BKZ 59 b	12576	6.37	900.00	12576	7.75	900.00	12576	7.59	900.00	12576	8.23	900.00
BKZ 510 b	12011	9.93	900.00	11955	10.62	900.00	12011	9.78	900.00	11955	10.19	900.00
BKZ 61 b	7600	11.78	900.00	7562	10.46	900.00	7600	9.16	900.00	7597	12.10	900.00
BKZ 62 b	6499	1.14	900.00	**6499**	**0.00**	**423.26**	6499	0.12	900.00	**6499**	**0.00**	**362.39**
BKZ 63 b	7030	3.28	900.00	**7021**	**0.00**	**900**	7030	3.66	900.00	7021	2.39	900.00
BKZ 64 b	7339	6.59	900.00	7301	3.41	900.00	7301	4.42	900.00	7299	4.20	900.00
BKZ 65 b	**6892**	**0.00**	**664.78**	**6892**	**0.00**	**155.26**	**6892**	**0.00**	**331.96**	**6892**	**0.00**	**107.78**
BKZ 41 c	13617	13.05	900.00	13617	15.32	900.00	13617	12.79	900.00	13617	15.93	900.00
BKZ 42 c	13619	13.81	900.00	13619	16.66	900.00	13926	15.11	900.00	13761	19.01	900.00
BKZ 43 c	14169	12.82	900.00	14110	13.52	900.00	14444	13.04	900.00	14073	11.56	900.00
BKZ 44 c	14774	14.61	900.00	14774	15.56	900.00	14774	12.73	900.00	14774	16.36	900.00
BKZ 45 c	15274	14.88	900.00	15255	18.86	900.00	15555	17.09	900.00	15256	17.86	900.00
BKZ 46 c	13334	11.48	900.00	13334	13.12	900.00	13334	11.84	900.00	13318	11.72	900.00
BKZ 47 c	13224	10.56	900.00	13157	11.55	900.00	13427	11.83	900.00	13180	10.93	900.00
BKZ 48 c	15046	11.81	900.00	15046	12.84	900.00	15406	13.45	900.00	15046	12.91	900.00
BKZ 49 c	14644	13.41	900.00	14644	16.00	900.00	14809	14.63	900.00	14523	13.98	900.00
BKZ 410 c	14459	14.67	900.00	14347	15.44	900.00	14459	13.63	900.00	14358	17.08	900.00
BKZ 51 c	10525	6.40	900.00	10525	6.73	900.00	10525	5.48	900.00	10525	7.77	900.00
BKZ 52 c	11653	8.49	900.00	11507	9.03	900.00	11753	9.32	900.00	11507	9.50	900.00
BKZ 53 c	10901	7.09	900.00	10844	5.92	900.00	11139	8.39	900.00	10841	6.00	900.00
BKZ 54 c*	9187	0.38	**900**	**9187**	**0.00**	**214.84**	**9187**	**0.00**	**557.38**	**9187**	**0.00**	**177.68**
BKZ 55 c	10965	4.55	900.00	10965	6.28	900.00	11088	4.83	900.00	10963	4.43	900.00
BKZ 56 c	12164	12.17	900.00	12164	11.49	900.00	12350	11.82	900.00	12164	11.99	900.00
BKZ 57 c	11590	7.41	900.00	11590	8.19	900.00	11590	5.97	900.00	11590	8.56	900.00
BKZ 58 c	11601	8.64	900.00	11541	9.91	900.00	11738	8.52	900.00	11551	7.88	900.00
BKZ 59 c	11949	11.07	900.00	11823	11.84	900.00	11949	10.36	900.00	11802	12.75	900.00
BKZ 510 c	11724	10.14	900.00	11724	12.90	900.00	11724	9.22	900.00	11724	12.09	900.00
BKZ 61 c	6800	2.07	900.00	**6794**	**0.00**	495.89	6800	1.31	900.00	**6794**	**0.00**	661.88
BKZ 62 c	**6366**	**0.00**	**859.64**	**6366**	**0.00**	**273.06**	**6366**	**0.00**	**837.26**	**6366**	**0.00**	**196.2**
BKZ 63 c	6940	3.77	900.00	**6940**	**0.00**	**675.42**	6940	2.25	900.00	**6940**	**0.00**	746.73
BKZ 64 c	7069	6.67	900.00	6951	2.34	900.00	6990	3.88	900.00	7123	3.37	900.00
BKZ 65 c	6746	3.34	900.00	**6746**	**0.00**	**678.74**	6757	1.78	900.00	**6746**	**0.00**	530.35
Average		9.04			9.45			8.87			9.48	

Table B.4: Comparison among the hybrid algorithms to the set BKZ.

	Max-max MCLP				*Max-min* MCLP			
Instance	w^u	$\psi^u(X^{\text{MMA}})$	$dev(w^u)$	T(s)	w^l	$\psi^l(X^{\text{MMI}})$	$dev(w^l)$	T(s)
BKZ 41	1002869	167062	16.66	0.01	499137	93823	18.80	0.01
BKZ 42	998640	166218	16.64	0.00	496362	93028	18.74	0.01
BKZ 43	1001423	168679	16.84	0.01	501480	93891	18.72	0.00
BKZ 44	1011543	168899	16.70	0.00	513102	93502	18.22	0.00
BKZ 45	1011246	167214	16.54	0.01	506243	93006	18.37	0.01
BKZ 46	984244	165297	16.79	0.00	493378	92829	18.81	0.01
BKZ 47	993330	166344	16.75	0.01	500351	93870	18.76	0.00
BKZ 48	1004250	169335	16.86	0.01	499791	93698	18.75	0.00
BKZ 49	1008918	168750	16.73	0.01	511758	94191	18.41	0.00
BKZ 410	982933	166237	16.91	0.01	485434	92971	19.15	0.01
BKZ 51	1983595	176103	8.88	0.01	1003722	96669	9.63	0.01
BKZ 52	1977715	176421	8.92	0.01	988314	97231	9.84	0.01
BKZ 53	1977813	174585	8.83	0.02	989868	96631	9.76	0.01
BKZ 54	1993920	174601	8.76	0.01	997991	96537	9.67	0.01
BKZ 55	1991505	175100	8.79	0.01	989236	96563	9.76	0.01
BKZ 56	1982904	178138	8.98	0.01	986848	96749	9.80	0.01
BKZ 57	2006562	175215	8.73	0.02	1010514	96830	9.58	0.01
BKZ 58	1979736	175514	8.87	0.02	989953	96292	9.73	0.01
BKZ 59	2003239	179324	8.95	0.02	1010305	96726	9.57	0.02
BKZ 510	1993937	175717	8.81	0.01	1001193	96264	9.61	0.01
BKZ 61	983672	169885	17.27	0.01	487287	95517	19.60	0.01
BKZ 62	1009658	172079	17.04	0.00	509191	94631	18.58	0.01
BKZ 63	1000747	169702	16.96	0.01	490675	94588	19.28	0.01
BKZ 64	1009930	170341	16.87	0.01	502857	94635	18.82	0.01
BKZ 65	998635	170670	17.09	0.01	513875	96019	18.69	0.01
BKZ 41 b	1007688	166982	16.57	0.01	514767	94686	18.39	0.00
BKZ 42 b	1014409	170137	16.77	0.00	519883	94135	18.11	0.01
BKZ 43 b	999533	165331	16.54	0.01	499227	92863	18.60	0.01
BKZ 44 b	1020511	170575	16.71	0.01	520603	94423	18.14	0.01
BKZ 45 b	994636	166467	16.74	0.01	493750	93231	18.88	0.01
BKZ 46 b	984922	169618	17.22	0.01	503115	94009	18.69	0.00
BKZ 47 b	986414	166320	16.86	0.01	485711	92259	18.99	0.01
BKZ 48 b	1018835	167530	16.44	0.00	509798	93025	18.25	0.01
BKZ 49 b	1010193	168147	16.65	0.01	506510	93592	18.48	0.00
BKZ 410 b	995629	168821	16.96	0.01	497950	94072	18.89	0.00
BKZ 51 b	2001782	178865	8.94	0.02	1013975	97251	9.59	0.01
BKZ 52 b	1958376	173762	8.87	0.01	968358	95550	9.87	0.01
BKZ 53 b	1991973	177595	8.92	0.01	1002099	97153	9.69	0.01
BKZ 54 b	1984834	176750	8.91	0.02	992151	96855	9.76	0.01
BKZ 55 b	1987908	175227	8.81	0.01	988038	96149	9.73	0.01
BKZ 56 b	1978781	178163	9.00	0.01	980827	96705	9.86	0.01
BKZ 57 b	1984524	175282	8.83	0.01	975268	96392	9.88	0.01
BKZ 58 b	2016429	175487	8.70	0.02	1000338	96577	9.65	0.01
BKZ 59 b	1976206	176657	8.94	0.02	982085	95913	9.77	0.01
BKZ 510 b	1997940	175817	8.80	0.02	994525	96331	9.69	0.01
BKZ 61 b	984772	165590	16.82	0.01	498826	94088	18.86	0.01
BKZ 62 b	999898	172169	17.22	0.01	504883	95237	18.86	0.01
BKZ 63 b	998381	170517	17.08	0.01	496413	94011	18.94	0.01
BKZ 64 b	997663	168753	16.91	0.01	500642	94221	18.82	0.01
BKZ 65 b	1011501	170296	16.84	0.01	501827	95227	18.98	0.01
BKZ 41 c	1007837	169712	16.84	0.00	513777	93988	18.29	0.01
BKZ 42 c	984986	163347	16.58	0.00	487183	93201	19.13	0.01
BKZ 43 c	1008328	166134	16.48	0.00	501210	91995	18.35	0.00
BKZ 44 c	1007809	168661	16.74	0.01	499209	93773	18.78	0.01
BKZ 45 c	994790	165771	16.66	0.01	502687	92745	18.45	0.01
BKZ 46 c	982950	168135	17.11	0.00	495468	93998	18.97	0.00
BKZ 47 c	992510	167070	16.83	0.01	501298	93797	18.71	0.01
BKZ 48 c	984920	164222	16.67	0.01	497535	92231	18.54	0.00
BKZ 49 c	1012720	166626	16.45	0.01	506099	92643	18.31	0.01
BKZ 410 c	1000482	169537	16.95	0.00	502169	93589	18.64	0.01
BKZ 51 c	2016180	176719	8.77	0.01	1010191	96522	9.55	0.00
BKZ 52 c	1977927	176020	8.90	0.02	990023	96713	9.77	0.01
BKZ 53 c	1999888	176612	8.83	0.01	985715	96747	9.81	0.01
BKZ 54 c	1996528	175631	8.80	0.01	995035	96308	9.68	0.01
BKZ 55 c	1993182	177460	8.90	0.02	1009973	97048	9.61	0.01
BKZ 56 c	1997327	177217	8.87	0.01	1005431	96828	9.63	0.01
BKZ 57 c	2011944	177447	8.82	0.02	1010224	96956	9.60	0.01
BKZ 58 c	2018286	178393	8.84	0.02	1017994	97161	9.54	0.01
BKZ 59 c	1976825	172905	8.75	0.02	990754	96671	9.76	0.01
BKZ 510 c	2045409	177559	8.68	0.02	1025118	96592	9.42	0.01
BKZ 61 c	1012012	172850	17.08	0.01	501990	95605	19.05	0.01
BKZ 62 c	1004397	171624	17.09	0.01	497378	95282	19.16	0.01
BKZ 63 c	1002609	167967	16.75	0.01	509989	94881	18.60	0.01
BKZ 64 c	1001894	170064	16.97	0.00	508606	94853	18.65	0.01
BKZ 65 c	975905	167605	17.17	0.01	489065	94722	19.37	0.01
Average			13.64				15.11	

Table B.5: Results of the *max-max* MCLP and the *max-min* MCLP for the set BKZ of instances with $T = 0.1 \times |M|$.

	Max-max MCLP				*Max-min* MCLP			
Instance	w^u	$\psi^u(X^{\text{MMA}})$	$dev(w^u)$	T(s)	w^l	$\psi^l(X^{\text{MMI}})$	$dev(w^l)$	T(s)
BKZ 41	1002869	315344	31.44	0.00	499137	179877	36.04	0.01
BKZ 42	998640	315203	31.56	0.01	496362	179526	36.17	0.01
BKZ 43	1001423	319177	31.87	0.01	501480	180549	36.00	0.00
BKZ 44	1011543	319315	31.57	0.00	513102	179563	35.00	0.00
BKZ 45	1011246	315480	31.20	0.01	506243	180000	35.56	0.01
BKZ 46	984244	312053	31.70	0.00	493378	177892	36.06	0.01
BKZ 47	993330	312889	31.50	0.01	500351	180201	36.01	0.01
BKZ 48	1004250	317732	31.64	0.00	499791	179395	35.89	0.00
BKZ 49	1008918	318535	31.57	0.00	511758	182092	35.58	0.00
BKZ 410	982933	314620	32.01	0.00	485434	178313	36.73	0.00
BKZ 51	1983595	339324	17.11	0.01	1003722	189257	18.86	0.01
BKZ 52	1977715	339584	17.17	0.02	988314	190288	19.25	0.01
BKZ 53	1977813	335456	16.96	0.01	989868	188705	19.06	0.01
BKZ 54	1993920	336911	16.90	0.02	997991	189478	18.99	0.01
BKZ 55	1991505	335692	16.86	0.01	989236	189206	19.13	0.01
BKZ 56	1982904	340323	17.16	0.02	986848	189562	19.21	0.01
BKZ 57	2006562	337360	16.81	0.01	1010514	190091	18.81	0.01
BKZ 58	1979736	339069	17.13	0.02	989953	188448	19.04	0.01
BKZ 59	2003239	344895	17.22	0.01	1010305	190109	18.82	0.01
BKZ 510	1993937	337727	16.94	0.01	1001193	189410	18.92	0.01
BKZ 61	983672	315261	32.05	0.01	487287	179784	36.89	0.01
BKZ 62	1009658	319559	31.65	0.00	509191	180736	35.49	0.01
BKZ 63	1000747	313002	31.28	0.01	490675	179751	36.63	0.01
BKZ 64	1009930	316210	31.31	0.01	502857	180358	35.87	0.01
BKZ 65	998635	316283	31.67	0.01	513875	183229	35.66	0.01
BKZ 41 b	1007688	313617	31.12	0.01	514767	181955	35.35	0.01
BKZ 42 b	1014409	319361	31.48	0.01	519883	181882	34.99	0.01
BKZ 43 b	999533	310231	31.04	0.01	499227	179590	35.97	0.01
BKZ 44 b	1020511	320281	31.38	0.01	520603	182112	34.98	0.01
BKZ 45 b	994636	315420	31.71	0.01	493750	179108	36.28	0.01
BKZ 46 b	984922	317461	32.23	0.01	503115	181233	36.02	0.01
BKZ 47 b	986414	315089	31.94	0.01	485711	177098	36.46	0.01
BKZ 48 b	1018835	315268	30.94	0.01	509798	177654	34.85	0.01
BKZ 49 b	1010193	318169	31.50	0.01	506510	179716	35.48	0.01
BKZ 410 b	995629	319877	32.13	0.01	497950	180785	36.31	0.00
BKZ 51 b	2001782	344259	17.20	0.02	1013975	191092	18.85	0.01
BKZ 52 b	1958376	335275	17.12	0.02	968358	187485	19.36	0.01
BKZ 53 b	1991973	342338	17.19	0.01	1002099	190638	19.02	0.01
BKZ 54 b	1984834	339157	17.09	0.02	992151	189974	19.15	0.01
BKZ 55 b	1987908	336141	16.91	0.01	988038	187799	19.01	0.01
BKZ 56 b	1978781	341405	17.25	0.01	980827	189241	19.29	0.01
BKZ 57 b	1984524	338032	17.03	0.02	975268	189513	19.43	0.01
BKZ 58 b	2016429	339434	16.83	0.02	1000338	189180	18.91	0.01
BKZ 59 b	1976206	340049	17.21	0.01	982085	188028	19.15	0.01
BKZ 510 b	1997940	338897	16.96	0.02	994525	189305	19.03	0.01
BKZ 61 b	984772	308093	31.29	0.01	498826	177027	35.49	0.01
BKZ 62 b	999898	319791	31.98	0.01	504883	180708	35.79	0.00
BKZ 63 b	998381	316219	31.67	0.01	496413	178718	36.00	0.01
BKZ 64 b	997663	314640	31.54	0.01	500642	177729	35.50	0.01
BKZ 65 b	1011501	317080	31.35	0.00	501827	181233	36.11	0.01
BKZ 41 c	1007837	319505	31.70	0.00	513777	181188	35.27	0.01
BKZ 42 c	984986	307663	31.24	0.01	487183	177150	36.36	0.01
BKZ 43 c	1008328	315244	31.26	0.00	501210	177369	35.39	0.01
BKZ 44 c	1007809	318461	31.60	0.01	499209	179856	36.03	0.01
BKZ 45 c	994790	313955	31.56	0.00	502687	177984	35.41	0.01
BKZ 46 c	982950	313185	31.86	0.00	495468	178379	36.00	0.01
BKZ 47 c	992510	313432	31.58	0.00	501298	179188	35.74	0.01
BKZ 48 c	984920	308102	31.28	0.00	497535	177205	35.62	0.01
BKZ 49 c	1012720	314822	31.09	0.01	506099	177789	35.13	0.00
BKZ 410 c	1000482	319512	31.94	0.01	502169	179821	35.81	0.01
BKZ 51 c	2016180	338396	16.78	0.02	1010191	189197	18.73	0.01
BKZ 52 c	1977927	336648	17.02	0.01	990023	189112	19.10	0.01
BKZ 53 c	1999888	340667	17.03	0.02	985715	188903	19.16	0.01
BKZ 54 c	1996528	338847	16.97	0.01	995035	189253	19.02	0.01
BKZ 55 c	1993182	340578	17.09	0.02	1009973	190692	18.88	0.01
BKZ 56 c	1997327	340224	17.03	0.01	1005431	190231	18.92	0.01
BKZ 57 c	2011944	341568	16.98	0.01	1010224	190305	18.84	0.01
BKZ 58 c	2018286	342287	16.96	0.01	1017994	190778	18.74	0.01
BKZ 59 c	1976825	333097	16.85	0.02	990754	189198	19.10	0.01
BKZ 510 c	2045409	341732	16.71	0.02	1025118	189750	18.51	0.01
BKZ 61 c	1012012	321534	31.77	0.01	501990	181641	36.18	0.01
BKZ 62 c	1004397	316967	31.56	0.01	497378	180293	36.25	0.01
BKZ 63 c	1002609	313757	31.29	0.01	509989	180115	35.32	0.01
BKZ 64 c	1001894	315818	31.52	0.01	508606	179369	35.27	0.01
BKZ 65 c	975905	309020	31.66	0.01	489065	179802	36.76	0.01
Average			25.74				29.09	

Table B.6: Results of the *max-max* MCLP and the *max-min* MCLP for the set BKZ of instances with $T = 0.2 \times |M|$.

	Max-max MCLP				*Max-min* MCLP			
Instance	w^u	$\psi^u(X^{\text{MMA}})$	$dev(w^u)$	T(s)	w^l	$\psi^l(X^{\text{MMI}})$	$dev(w^l)$	T(s)
BKZ 41	1002869	446456	44.52	0.01	499137	255273	51.14	0.01
BKZ 42	998640	445631	44.62	0.01	496362	253117	50.99	0.01
BKZ 43	1001423	450856	45.02	0.01	501480	256475	51.14	0.00
BKZ 44	1011543	450192	44.51	0.00	513102	255858	49.86	0.00
BKZ 45	1011246	445526	44.06	0.00	506243	256535	50.67	0.01
BKZ 46	984244	441040	44.81	0.00	493378	252922	51.26	0.01
BKZ 47	993330	442201	44.52	0.01	500351	255376	51.04	0.01
BKZ 48	1004250	447006	44.51	0.01	499791	254139	50.85	0.01
BKZ 49	1008918	448969	44.50	0.00	511758	259367	50.68	0.01
BKZ 410	982933	441931	44.96	0.01	485434	253120	52.14	0.01
BKZ 51	1983595	489848	24.69	0.01	1003722	276804	27.58	0.01
BKZ 52	1977715	487894	24.67	0.02	988314	277369	28.06	0.01
BKZ 53	1977813	482274	24.38	0.01	989868	274741	27.76	0.01
BKZ 54	1993920	486630	24.41	0.02	997991	276590	27.71	0.01
BKZ 55	1991505	484716	24.34	0.02	989236	276557	27.96	0.01
BKZ 56	1982904	488824	24.65	0.01	986848	277010	28.07	0.01
BKZ 57	2006562	486487	24.24	0.02	1010514	277906	27.50	0.01
BKZ 58	1979736	489055	24.70	0.01	989953	275010	27.78	0.01
BKZ 59	2003239	497207	24.82	0.01	1010305	278498	27.57	0.01
BKZ 510	1993937	486832	24.42	0.02	1001193	276761	27.64	0.01
BKZ 61	983672	444892	45.23	0.01	487287	253738	52.07	0.01
BKZ 62	1009658	449545	44.52	0.01	509191	257183	50.51	0.01
BKZ 63	1000747	440863	44.05	0.01	490675	255033	51.98	0.01
BKZ 64	1009930	445951	44.16	0.01	502857	256178	50.94	0.01
BKZ 65	998635	444795	44.54	0.01	513875	260041	50.60	0.01
BKZ 41 b	1007688	443196	43.98	0.01	514767	259348	50.38	0.00
BKZ 42 b	1014409	450119	44.37	0.01	519883	258420	49.71	0.01
BKZ 43 b	999533	439996	44.02	0.01	499227	255273	51.13	0.00
BKZ 44 b	1020511	452321	44.32	0.00	520603	259531	49.85	0.00
BKZ 45 b	994636	444645	44.70	0.01	493750	254191	51.48	0.01
BKZ 46 b	984922	445322	45.21	0.00	503115	256746	51.03	0.01
BKZ 47 b	986414	443809	44.99	0.00	485711	251279	51.73	0.01
BKZ 48 b	1018835	446668	43.84	0.00	509798	252765	49.58	0.01
BKZ 49 b	1010193	448830	44.43	0.01	506510	254591	50.26	0.01
BKZ 410 b	995629	451410	45.34	0.01	497950	256518	51.51	0.00
BKZ 51 b	2001782	494914	24.72	0.01	1013975	279537	27.57	0.01
BKZ 52 b	1958376	483560	24.69	0.02	968358	273966	28.29	0.01
BKZ 53 b	1991973	492737	24.74	0.02	1002099	278990	27.84	0.01
BKZ 54 b	1984834	487547	24.56	0.01	992151	277016	27.92	0.01
BKZ 55 b	1987908	485533	24.42	0.01	988038	273505	27.68	0.01
BKZ 56 b	1978781	491899	24.86	0.01	980827	275903	28.13	0.01
BKZ 57 b	1984524	487654	24.57	0.01	975268	277203	28.42	0.01
BKZ 58 b	2016429	491851	24.39	0.02	1000338	277137	27.70	0.01
BKZ 59 b	1976206	489482	24.77	0.01	982085	274594	27.96	0.01
BKZ 510 b	1997940	489303	24.49	0.02	994525	277136	27.87	0.01
BKZ 61 b	984772	435907	44.26	0.01	498826	250898	50.30	0.01
BKZ 62 b	999898	449533	44.96	0.01	504883	257434	50.99	0.01
BKZ 63 b	998381	444445	44.52	0.01	496413	252699	50.90	0.01
BKZ 64 b	997663	445011	44.61	0.01	500642	250501	50.04	0.01
BKZ 65 b	1011501	448221	44.31	0.01	501827	256627	51.14	0.00
BKZ 41 c	1007837	451995	44.85	0.01	513777	257713	50.16	0.00
BKZ 42 c	984986	434420	44.10	0.00	487183	250665	51.45	0.01
BKZ 43 c	1008328	444149	44.05	0.01	501210	251932	50.26	0.01
BKZ 44 c	1007809	448906	44.54	0.00	499209	255874	51.26	0.01
BKZ 45 c	994790	443972	44.63	0.01	502687	252742	50.28	0.01
BKZ 46 c	982950	438979	44.66	0.01	495468	253386	51.14	0.01
BKZ 47 c	992510	442780	44.61	0.00	501298	254483	50.76	0.00
BKZ 48 c	984920	434856	44.15	0.01	497535	252010	50.65	0.01
BKZ 49 c	1012720	445003	43.94	0.00	506099	253326	50.05	0.01
BKZ 410 c	1000482	449628	44.94	0.01	502169	256123	51.00	0.00
BKZ 51 c	2016180	488122	24.21	0.01	1010191	276468	27.37	0.01
BKZ 52 c	1977927	484360	24.49	0.01	990023	275401	27.82	0.01
BKZ 53 c	1999888	491850	24.59	0.02	985715	275693	27.97	0.01
BKZ 54 c	1996528	488899	24.49	0.01	995035	276429	27.78	0.01
BKZ 55 c	1993182	489204	24.54	0.01	1009973	279055	27.63	0.01
BKZ 56 c	1997327	490097	24.54	0.02	1005431	278144	27.66	0.01
BKZ 57 c	2011944	492434	24.48	0.01	1010224	278618	27.58	0.01
BKZ 58 c	2018286	492633	24.41	0.02	1017994	279327	27.44	0.01
BKZ 59 c	1976825	481409	24.35	0.01	990754	275792	27.84	0.01
BKZ 510 c	2045409	493919	24.15	0.02	1025118	277311	27.05	0.01
BKZ 61 c	1012012	453761	44.84	0.01	501990	256965	51.19	0.01
BKZ 62 c	1004397	447037	44.51	0.01	497378	254666	51.20	0.01
BKZ 63 c	1002609	443356	44.22	0.01	509989	255018	50.00	0.01
BKZ 64 c	1001894	444744	44.39	0.01	508606	255818	50.30	0.01
BKZ 65 c	975905	435422	44.62	0.01	489065	253583	51.85	0.01
Average			36.52				41.61	

Table B.7: Results of the *max-max* MCLP and the *max-min* MCLP for the set BKZ of instances with $T = 0.3 \times |M|$.

Instance	BLD				EB				B&C			
	$\rho(X)$	Gap	T (s)	Cuts	$\rho(X)$	Gap	T (s)	Cuts	$\rho(X)$	Gap	T (s)	Cuts
BKZ 41	58447	63.62	3600.00	9	58447	63.88	3600.00	172	58447	60.54	3600.00	4206
BKZ 42	58448	63.66	3600.00	7	58448	62.79	3600.00	154	58448	59.70	3600.00	4655
BKZ 43	58777	64.63	3600.00	8	58777	65.61	3600.00	166	58777	62.86	3600.00	4563
BKZ 44	59553	66.80	3600.00	7	59553	66.01	3600.00	78	59553	64.51	3600.00	4164
BKZ 45	59176	66.22	3600.00	7	59176	64.35	3600.00	186	59176	60.87	3600.00	4261
BKZ 46	60106	67.35	3600.00	7	60106	66.32	3600.00	152	60106	67.37	3600.00	4700
BKZ 47	55839	60.61	3600.00	10	55839	61.89	3600.00	130	55839	58.66	3600.00	3825
BKZ 48	59524	65.49	3600.00	8	59524	65.13	3600.00	148	59524	62.01	3600.00	4271
BKZ 49	59170	62.95	3600.00	9	59170	63.99	3600.00	72	59170	58.80	3600.00	4476
BKZ 410	59555	68.30	3600.00	8	59555	67.67	3600.00	139	59555	65.98	3600.00	4782
BKZ 51	69422	69.52	3600.00	7	69422	69.10	3600.00	165	69422	67.40	3600.00	4932
BKZ 52	69670	72.08	3600.00	6	69670	70.03	3600.00	98	69670	68.84	3600.00	5126
BKZ 53	67493	66.61	3600.00	7	67493	67.21	3600.00	156	67493	64.07	3600.00	4949
BKZ 54	69199	67.08	3600.00	8	69199	68.85	3600.00	109	69199	66.44	3600.00	4867
BKZ 55	67127	67.96	3600.00	6	67127	67.79	3600.00	197	67127	66.45	3600.00	5567
BKZ 56	68944	71.34	3600.00	8	68944	70.69	3600.00	79	68944	67.12	3600.00	4940
BKZ 57	68477	68.03	3600.00	8	68477	66.20	3600.00	163	68477	64.27	3600.00	4844
BKZ 58	71281	75.43	3600.00	8	71281	74.15	3600.00	170	71281	70.90	3600.00	5185
BKZ 59	72181	75.25	3600.00	8	72181	74.67	3600.00	123	72181	73.12	3600.00	5375
BKZ 510	71039	74.40	3600.00	8	71039	73.40	3600.00	124	71039	72.50	3600.00	5286
BKZ 61	57985	65.31	3600.00	7	57985	66.03	3600.00	187	57985	67.43	3600.00	4872
BKZ 62	59691	65.17	3600.00	7	59691	64.98	3600.00	131	59691	68.59	3600.00	5738
BKZ 63	56383	63.77	3600.00	6	56383	63.76	3600.00	147	56383	66.12	3600.00	5835
BKZ 64	58339	64.11	3600.00	7	58339	63.91	3600.00	94	58339	66.88	3600.00	5293
BKZ 65	56454	63.14	3600.00	7	56454	62.84	3600.00	154	56454	66.09	3600.00	4958
BKZ 41 b	56969	63.25	3600.00	8	56969	62.18	3600.00	124	56969	62.59	3600.00	4771
BKZ 42 b	58449	64.57	3600.00	8	58449	62.44	3600.00	172	58449	62.31	3600.00	4415
BKZ 43 b	56346	64.49	3600.00	7	56346	63.26	3600.00	140	56346	60.94	3600.00	4700
BKZ 44 b	58827	64.41	3600.00	8	58827	63.47	3600.00	139	58827	62.81	3600.00	4221
BKZ 45 b	57769	62.73	3600.00	8	57769	62.89	3600.00	153	57769	61.30	3600.00	5133
BKZ 46 b	57481	65.31	3600.00	8	57481	65.46	3600.00	102	57481	63.90	3600.00	4570
BKZ 47 b	60551	63.91	3600.00	9	60551	64.80	3600.00	190	60551	62.81	3600.00	4410
BKZ 48 b	59397	65.35	3600.00	8	59397	65.65	3600.00	70	59397	63.07	3600.00	4540
BKZ 49 b	61517	67.35	3600.00	8	61517	67.31	3600.00	78	61517	66.11	3600.00	4592
BKZ 410 b	60507	66.57	3600.00	7	60507	65.99	3600.00	129	60507	63.40	3600.00	4343
BKZ 51 b	70463	69.93	3600.00	8	70463	70.83	3600.00	157	70463	69.80	3600.00	5243
BKZ 52 b	70386	71.43	3600.00	7	70386	69.14	3600.00	193	70386	66.11	3600.00	4493
BKZ 53 b	70100	70.67	3600.00	7	70100	68.92	3600.00	73	70100	67.84	3600.00	5159
BKZ 54 b	69006	71.13	3600.00	8	69006	71.99	3600.00	172	69006	68.76	3600.00	4719
BKZ 55 b	69782	67.90	3600.00	9	69782	68.64	3600.00	199	69782	68.76	3600.00	5637
BKZ 56 b	70959	74.23	3600.00	6	70959	72.10	3600.00	194	70959	68.60	3600.00	5162
BKZ 57 b	70045	72.82	3600.00	7	70045	71.82	3600.00	166	70045	70.87	3600.00	5504
BKZ 58 b	70982	70.33	3600.00	8	70982	70.03	3600.00	150	70982	69.25	3600.00	5022
BKZ 59 b	71001	73.27	3600.00	7	71001	71.40	3600.00	171	71001	70.40	3600.00	5328
BKZ 510 b	70168	72.45	3600.00	8	70168	71.96	3600.00	164	70168	71.22	3600.00	5312
BKZ 61 b	55442	63.35	3600.00	7	55442	63.68	3600.00	119	55442	64.87	3600.00	4947
BKZ 62 b	59867	67.81	3600.00	7	59867	68.64	3600.00	181	59867	70.84	3600.00	5116
BKZ 63 b	59276	65.57	3600.00	8	59276	67.50	3600.00	199	59276	68.61	3600.00	5453
BKZ 64 b	61014	67.43	3600.00	8	61014	68.86	3600.00	165	61014	71.13	3600.00	5052
BKZ 65 b	59575	65.35	3600.00	7	59575	65.10	3600.00	178	59575	68.89	3600.00	5321
BKZ 41 c	60156	67.39	3600.00	8	60156	67.74	3600.00	140	60156	64.74	3600.00	4497
BKZ 42 c	57219	64.99	3600.00	9	57219	65.14	3600.00	110	57219	63.42	3600.00	3882
BKZ 43 c	60069	64.40	3600.00	8	60069	64.30	3600.00	138	60069	62.66	3600.00	4763
BKZ 44 c	59179	67.01	3600.00	8	59179	67.07	3600.00	198	59179	64.43	3600.00	4128
BKZ 45 c	58937	65.81	3600.00	8	58937	66.39	3600.00	105	58937	65.01	3600.00	4201
BKZ 46 c	56852	63.96	3600.00	8	56852	64.29	3600.00	124	56852	65.26	3600.00	4721
BKZ 47 c	56004	62.32	3600.00	7	56004	62.26	3600.00	158	56004	59.12	3600.00	4505
BKZ 48 c	56544	61.70	3600.00	9	56544	61.15	3600.00	125	56544	60.95	3600.00	4479
BKZ 49 c	59412	65.81	3600.00	8	59412	64.36	3600.00	189	59412	63.87	3600.00	4645
BKZ 410 c	59444	65.75	3600.00	8	59444	65.75	3600.00	146	59444	61.95	3600.00	4212
BKZ 51 c	69854	75.00	3600.00	6	69854	71.60	3600.00	88	69854	69.06	3600.00	4843
BKZ 52 c	67593	68.94	3600.00	7	67593	67.46	3600.00	147	67593	66.68	3600.00	5474
BKZ 53 c	71241	73.91	3600.00	6	71241	70.90	3600.00	152	71241	68.38	3600.00	5179
BKZ 54 c	69988	69.71	3600.00	8	69988	68.86	3600.00	115	69988	65.47	3600.00	5158
BKZ 55 c	67983	69.21	3600.00	8	67983	70.68	3600.00	78	67983	65.10	3600.00	5115
BKZ 56 c	70661	74.23	3600.00	7	70661	75.21	3600.00	95	70661	71.21	3600.00	5451
BKZ 57 c	71148	72.06	3600.00	7	71148	71.17	3600.00	137	71148	68.42	3600.00	5086
BKZ 58 c	69949	70.67	3600.00	7	69949	70.06	3600.00	101	69949	70.49	3600.00	5260
BKZ 59 c	68918	72.18	3600.00	8	68918	72.93	3600.00	132	68918	69.29	3600.00	4812
BKZ 510 c	73105	74.72	3600.00	6	73105	73.53	3600.00	197	73105	73.30	3600.00	5105
BKZ 61 c	58522	63.91	3600.00	7	58522	64.22	3600.00	148	58522	66.52	3600.00	5666
BKZ 62 c	56937	62.21	3600.00	8	56937	62.64	3600.00	102	56937	65.27	3600.00	5103
BKZ 63 c	59309	67.12	3600.00	7	59309	68.09	3600.00	80	59309	69.23	3600.00	5492
BKZ 64 c	57540	64.31	3600.00	8	57540	64.70	3600.00	123	57540	66.71	3600.00	5075
BKZ 65 c	53612	59.25	3600.00	7	53612	60.50	3600.00	106	53612	61.74	3600.00	4584
Average		67.43				67.15				66.01		

Table B.8: Comparison among the exact algorithms to the set BKZ for $T = 0.1 \times |M|$.

Instance	BLD				EB				B&C			
	$\rho(X)$	Gap	T (s)	Cuts	$\rho(X)$	Gap	T (s)	Cuts	$\rho(X)$	Gap	T (s)	Cuts
BKZ 41	81910	57.02	3600.00	6	81910	57.69	3600.00	77	81910	66.32	3600.00	2653
BKZ 42	83679	58.07	3600.00	7	83679	58.36	3600.00	161	83679	68.71	3600.00	1326
BKZ 43	80563	54.88	3600.00	7	80563	55.06	3600.00	54	80563	63.03	3600.00	1307
BKZ 44	82706	55.47	3600.00	7	82706	55.68	3600.00	155	82706	64.82	3600.00	1100
BKZ 45	84004	56.45	3600.00	7	84004	58.46	3600.00	116	84004	67.85	3600.00	1333
BKZ 46	82552	54.62	3600.00	7	82552	54.93	3600.00	74	82552	65.38	3600.00	922
BKZ 47	80877	55.40	3600.00	7	80877	57.40	3600.00	78	80877	64.32	3600.00	2145
BKZ 48	83906	56.44	3600.00	7	83906	58.13	3600.00	159	83906	66.63	3600.00	1618
BKZ 49	81427	55.90	3600.00	7	81427	57.91	3600.00	168	81427	66.05	3600.00	130
BKZ 410	77683	53.33	3600.00	8	77683	54.13	3600.00	150	77683	64.11	3600.00	1137
BKZ 51	116839	65.34	3600.00	7	116839	66.99	3600.00	138	116839	76.42	3600.00	3479
BKZ 52	113746	63.26	3600.00	7	113746	63.50	3600.00	117	113746	73.17	3600.00	3255
BKZ 53	112937	62.96	3600.00	7	112937	64.40	3600.00	127	112937	71.53	3600.00	2785
BKZ 54	117507	66.00	3600.00	7	117507	67.58	3600.00	141	117507	74.76	3600.00	3218
BKZ 55	115021	64.94	3600.00	8	115021	67.18	3600.00	159	115021	75.20	3600.00	2809
BKZ 56	110857	61.78	3600.00	6	110857	63.40	3600.00	147	110857	69.98	3600.00	3021
BKZ 57	115320	63.77	3600.00	7	115320	64.08	3600.00	102	115320	74.24	3600.00	3163
BKZ 58	115598	64.25	3600.00	7	115598	64.90	3600.00	87	115598	73.19	3600.00	2768
BKZ 59	115578	63.74	3600.00	7	115578	64.21	3600.00	74	115578	70.81	3600.00	2866
BKZ 510	115851	64.85	3600.00	7	115851	65.04	3600.00	55	115851	75.35	3600.00	3141
BKZ 61	79185	54.24	3600.00	6	79185	56.00	3600.00	72	79185	63.36	3600.00	1784
BKZ 62	81344	53.62	3600.00	7	81344	56.07	3600.00	112	81344	64.07	3600.00	3226
BKZ 63	79315	53.11	3600.00	6	79315	52.70	3600.00	81	79315	62.87	3600.00	2578
BKZ 64	83163	56.88	3600.00	7	83163	57.71	3600.00	88	83163	66.07	3600.00	3335
BKZ 65	74905	54.43	3600.00	8	74905	54.79	3600.00	130	74905	62.75	3600.00	3009
BKZ 41 b	80312	55.36	3600.00	8	80312	56.00	3600.00	160	80312	64.28	3600.00	1616
BKZ 42 b	80717	55.32	3600.00	6	80717	55.71	3600.00	123	80717	64.83	3600.00	1212
BKZ 43 b	83095	55.75	3600.00	7	83095	57.11	3600.00	53	83095	64.18	3600.00	1396
BKZ 44 b	83034	54.91	3600.00	7	83034	56.49	3600.00	58	83034	64.49	3600.00	1862
BKZ 45 b	79188	54.10	3600.00	7	79188	54.37	3600.00	109	79188	61.58	3600.00	1917
BKZ 46 b	74510	51.81	3600.00	7	74510	52.19	3600.00	94	74510	57.83	3600.00	1533
BKZ 47 b	79918	55.14	3600.00	7	79918	55.52	3600.00	148	79918	65.29	3600.00	826
BKZ 48 b	85962	58.78	3600.00	8	85962	58.06	3600.00	169	85962	68.28	3600.00	1329
BKZ 49 b	86290	56.84	3600.00	6	86290	57.27	3600.00	53	86290	63.51	3600.00	1784
BKZ 410 b	80130	54.52	3600.00	7	80130	54.84	3600.00	164	80130	63.74	3600.00	1783
BKZ 51 b	113566	62.59	3600.00	8	113566	62.76	3600.00	78	113566	73.51	3600.00	2972
BKZ 52 b	114953	65.24	3600.00	8	114953	66.90	3600.00	87	114953	73.41	3600.00	3063
BKZ 53 b	114305	64.12	3600.00	7	114305	64.31	3600.00	71	114305	73.83	3600.00	3276
BKZ 54 b	112282	63.49	3600.00	8	112282	64.06	3600.00	118	112282	73.14	3600.00	2919
BKZ 55 b	118668	66.17	3600.00	7	118668	66.32	3600.00	63	118668	74.57	3600.00	2854
BKZ 56 b	118053	65.97	3600.00	8	118053	67.49	3600.00	166	118053	75.96	3600.00	3325
BKZ 57 b	113942	62.91	3600.00	7	113942	63.74	3600.00	71	113942	72.23	3600.00	3231
BKZ 58 b	120248	67.90	3600.00	7	120248	68.74	3600.00	60	120248	76.03	3600.00	3323
BKZ 59 b	115389	63.84	3600.00	7	115389	65.28	3600.00	64	115389	73.18	3600.00	3055
BKZ 510 b	116495	65.40	3600.00	6	116495	65.94	3600.00	54	116495	75.23	3600.00	3331
BKZ 61 b	78572	56.68	3600.00	6	78572	58.27	3600.00	118	78572	67.38	3600.00	3312
BKZ 62 b	78369	52.87	3600.00	7	78369	54.52	3600.00	97	78369	62.55	3600.00	3558
BKZ 63 b	78510	53.67	3600.00	6	78510	53.19	3600.00	104	78510	61.25	3600.00	2627
BKZ 64 b	79162	53.64	3600.00	7	84845	57.56	3600.00	94	84845	65.47	3600.00	3593
BKZ 65 b	75407	54.30	3600.00	6	85815	58.14	3600.00	137	85815	66.24	3600.00	3208
BKZ 41 c	83162	56.68	3600.00	7	83162	57.07	3600.00	79	83162	65.26	3600.00	1731
BKZ 42 c	81866	55.81	3600.00	8	81866	56.97	3600.00	144	81866	65.20	3600.00	1369
BKZ 43 c	83110	54.89	3600.00	8	83110	56.37	3600.00	127	83110	61.85	3600.00	1952
BKZ 44 c	82259	55.46	3600.00	6	82259	56.42	3600.00	134	82259	65.24	3600.00	1881
BKZ 45 c	82734	55.33	3600.00	8	82734	55.83	3600.00	68	82734	65.42	3600.00	578
BKZ 46 c	75550	53.96	3600.00	7	75550	54.46	3600.00	134	75550	61.73	3600.00	1712
BKZ 47 c	79937	55.40	3600.00	7	79937	55.61	3600.00	127	79937	63.88	3600.00	1578
BKZ 48 c	79368	55.41	3600.00	7	79368	57.05	3600.00	137	79368	64.28	3600.00	2184
BKZ 49 c	84649	56.89	3600.00	7	84649	58.75	3600.00	144	84649	65.27	3600.00	1701
BKZ 410 c	81534	54.24	3600.00	9	81534	55.20	3600.00	71	81534	62.13	3600.00	1214
BKZ 51 c	116493	66.45	3600.00	6	116493	66.07	3600.00	126	116493	74.83	3600.00	3140
BKZ 52 c	111826	62.86	3600.00	7	111826	63.09	3600.00	79	111826	71.78	3600.00	2721
BKZ 53 c	117257	63.84	3600.00	8	117257	65.65	3600.00	66	117257	74.00	3600.00	3101
BKZ 54 c	117653	63.89	3600.00	8	117653	64.65	3600.00	150	117653	73.86	3600.00	2852
BKZ 55 c	110144	60.91	3600.00	7	110144	61.13	3600.00	126	110144	70.36	3600.00	2988
BKZ 56 c	113897	63.00	3600.00	8	113897	63.63	3600.00	135	113897	72.35	3600.00	2921
BKZ 57 c	116775	66.05	3600.00	7	116775	70.54	3600.00	54	116775	75.41	3600.00	3059
BKZ 58 c	114538	62.32	3600.00	8	114538	64.46	3600.00	129	114538	71.95	3600.00	2908
BKZ 59 c	114481	64.31	3600.00	7	114481	65.69	3600.00	133	114481	73.45	3600.00	3167
BKZ 510 c	120370	67.25	3600.00	7	120370	68.51	3600.00	70	120370	74.57	3600.00	2702
BKZ 61 c	82234	55.18	3600.00	7	82234	55.66	3600.00	103	82234	62.59	3600.00	2297
BKZ 62 c	80523	54.65	3600.00	6	80523	55.43	3600.00	103	80523	63.90	3600.00	3342
BKZ 63 c	83558	55.79	3600.00	6	83558	56.75	3600.00	116	83558	65.81	3600.00	3199
BKZ 64 c	79162	53.64	3600.00	7	79162	55.29	3600.00	72	79162	63.07	3600.00	2988
BKZ 65 c	75407	54.30	3600.00	6	75407	53.98	3600.00	54	75407	62.10	3600.00	3282
Average		58.81				59.80				68.04		

Table B.9: Comparison among the exact algorithms to the set BKZ for $T = 0.2 \times |M|$.

	BLD				EB				B&C			
Instance	$\rho(X)$	Gap	T (s)	Cuts	$\rho(X)$	Gap	T (s)	Cuts	$\rho(X)$	Gap	T (s)	Cuts
BKZ 41	84057	53.35	3600.00	8	84057	55.78	3600.00	67	84057	58.93	3600.00	855
BKZ 42	86949	58.16	3600.00	6	86949	59.06	3600.00	130	86949	66.51	3600.00	990
BKZ 43	81116	53.30	3600.00	6	81116	54.40	3600.00	129	81116	58.50	3600.00	962
BKZ 44	84919	50.83	3600.00	7	84919	52.48	3600.00	124	84919	58.01	3600.00	503
BKZ 45	88603	54.08	3600.00	7	88603	54.44	3600.00	86	88603	60.21	3600.00	746
BKZ 46	85543	52.52	3600.00	7	85543	53.07	3600.00	64	85543	63.82	3600.00	181
BKZ 47	83015	54.63	3600.00	6	83015	55.67	3600.00	118	83015	58.85	3600.00	647
BKZ 48	86638	53.34	3600.00	7	86638	55.19	3600.00	89	86638	60.01	3600.00	641
BKZ 49	80300	54.01	3600.00	6	80300	53.60	3600.00	120	80300	57.33	3600.00	736
BKZ 410	78350	51.33	3600.00	7	78350	51.74	3600.00	86	78350	58.51	3600.00	1026
BKZ 51	146862	59.42	3600.00	7	146862	59.95	3600.00	158	146862	69.77	3600.00	2542
BKZ 52	142741	57.09	3600.00	7	142741	57.63	3600.00	103	142741	66.50	3600.00	2672
BKZ 53	145054	58.14	3600.00	8	145054	60.50	3600.00	51	145054	68.83	3600.00	2600
BKZ 54	147638	59.88	3600.00	7	147638	61.46	3600.00	138	147638	70.27	3600.00	2788
BKZ 55	146392	58.57	3600.00	7	146392	60.32	3600.00	125	146392	68.00	3600.00	2482
BKZ 56	140342	56.79	3600.00	7	140342	58.80	3600.00	163	140342	66.29	3600.00	2747
BKZ 57	148061	58.50	3600.00	8	148061	61.07	3600.00	65	148061	70.46	3600.00	2203
BKZ 58	142878	58.16	3600.00	7	142878	60.09	3600.00	123	142878	66.45	3600.00	2768
BKZ 59	140573	56.20	3600.00	8	140573	58.30	3600.00	150	140573	65.12	3600.00	2422
BKZ 510	144963	58.69	3600.00	9	144963	59.91	3600.00	93	144963	71.61	3600.00	2971
BKZ 61	80727	52.16	3600.00	7	80727	52.65	3600.00	162	80727	59.86	3600.00	1303
BKZ 62	83328	52.80	3600.00	6	83328	53.80	3600.00	80	83328	59.38	3600.00	1689
BKZ 63	83817	52.69	3600.00	6	83817	53.77	3600.00	97	83817	60.13	3600.00	1265
BKZ 64	84911	54.61	3600.00	6	84911	55.62	3600.00	65	84911	60.92	3600.00	1471
BKZ 65	76058	52.43	3600.00	7	76058	53.02	3600.00	59	76058	59.21	3600.00	1200
BKZ 41 b	83109	53.68	3600.00	7	83109	54.10	3600.00	144	83109	62.58	3600.00	757
BKZ 42 b	83346	55.44	3600.00	7	83346	57.24	3600.00	105	83346	60.80	3600.00	793
BKZ 43 b	86101	55.61	3600.00	6	86101	56.39	3600.00	114	86101	61.02	3600.00	397
BKZ 44 b	84309	54.04	3600.00	7	84309	54.65	3600.00	75	84309	60.62	3600.00	871
BKZ 45 b	82577	51.57	3600.00	8	82577	52.52	3600.00	88	82577	57.29	3600.00	970
BKZ 46 b	75800	53.68	3600.00	7	75800	55.35	3600.00	54	75800	59.84	3600.00	1034
BKZ 47 b	79865	51.10	3600.00	6	79865	51.91	3600.00	84	79865	55.63	3600.00	822
BKZ 48 b	91065	55.66	3600.00	6	91065	55.26	3600.00	126	91065	62.86	3600.00	892
BKZ 49 b	88552	54.78	3600.00	6	88552	55.42	3600.00	51	88552	61.12	3600.00	846
BKZ 410 b	79189	52.42	3600.00	7	79189	52.82	3600.00	113	79189	64.50	3600.00	112
BKZ 51 b	142540	55.84	3600.00	8	142540	58.06	3600.00	107	142540	67.09	3600.00	2369
BKZ 52 b	141960	57.49	3600.00	8	141960	59.68	3600.00	81	141960	66.24	3600.00	2231
BKZ 53 b	139088	57.25	3600.00	6	139088	58.17	3600.00	97	139088	67.34	3600.00	2373
BKZ 54 b	141161	56.40	3600.00	9	141161	57.71	3600.00	114	141161	65.82	3600.00	2687
BKZ 55 b	149953	59.45	3600.00	8	149953	60.47	3600.00	89	149953	70.71	3600.00	2722
BKZ 56 b	146290	58.67	3600.00	8	146290	59.63	3600.00	95	146290	71.24	3600.00	2861
BKZ 57 b	143834	57.00	3600.00	8	143834	59.17	3600.00	109	143834	67.70	3600.00	2521
BKZ 58 b	150488	58.31	3600.00	8	150488	60.42	3600.00	140	150488	68.55	3600.00	2819
BKZ 59 b	144310	59.45	3600.00	6	144310	60.30	3600.00	81	144310	69.87	3600.00	2349
BKZ 510 b	145583	57.87	3600.00	7	145583	58.21	3600.00	101	145583	67.12	3600.00	2164
BKZ 61 b	79698	55.49	3600.00	6	79698	56.59	3600.00	148	79698	61.90	3600.00	1159
BKZ 62 b	77616	51.72	3600.00	7	77616	54.14	3600.00	55	77616	58.67	3600.00	1190
BKZ 63 b	82638	52.47	3600.00	6	82638	52.09	3600.00	133	82638	60.47	3600.00	2084
BKZ 64 b	86076	52.67	3600.00	7	86076	53.06	3600.00	89	86076	60.37	3600.00	1983
BKZ 65 b	88496	52.77	3600.00	7	88496	53.06	3600.00	147	88496	57.83	3600.00	911
BKZ 41 c	81126	53.26	3600.00	7	81126	53.90	3600.00	130	81126	58.62	3600.00	982
BKZ 42 c	88322	55.77	3600.00	6	88322	55.31	3600.00	154	88322	63.68	3600.00	654
BKZ 43 c	86852	52.55	3600.00	6	86852	52.22	3600.00	51	86852	60.39	3600.00	720
BKZ 44 c	86673	54.85	3600.00	7	86673	56.83	3600.00	153	86673	59.58	3600.00	932
BKZ 45 c	86569	52.86	3600.00	7	86569	53.35	3600.00	53	86569	58.35	3600.00	691
BKZ 46 c	77894	51.83	3600.00	7	77894	53.60	3600.00	72	77894	58.46	3600.00	866
BKZ 47 c	82091	55.67	3600.00	7	82091	57.76	3600.00	127	82091	63.49	3600.00	938
BKZ 48 c	83589	55.50	3600.00	7	83589	57.20	3600.00	138	83589	64.60	3600.00	1025
BKZ 49 c	88519	54.54	3600.00	7	88519	54.96	3600.00	148	88519	61.91	3600.00	563
BKZ 410 c	82546	52.87	3600.00	6	82546	53.70	3600.00	76	82546	56.71	3600.00	827
BKZ 51 c	146996	58.72	3600.00	7	146996	60.50	3600.00	163	146996	67.92	3600.00	2132
BKZ 52 c	140764	56.90	3600.00	8	140764	57.87	3600.00	145	140764	67.08	3600.00	2289
BKZ 53 c	145415	56.07	3600.00	8	145415	57.07	3600.00	145	145415	67.33	3600.00	2565
BKZ 54 c	148148	58.88	3600.00	7	148148	59.29	3600.00	164	148148	69.38	3600.00	2311
BKZ 55 c	139314	55.94	3600.00	7	139314	57.89	3600.00	98	139314	65.43	3600.00	2965
BKZ 56 c	142511	57.40	3600.00	7	142511	59.16	3600.00	141	142511	68.02	3600.00	2500
BKZ 57 c	143687	58.49	3600.00	7	143687	58.95	3600.00	88	143687	69.10	3600.00	2526
BKZ 58 c	143999	57.28	3600.00	8	143999	58.13	3600.00	89	143999	67.40	3600.00	2533
BKZ 59 c	145520	58.53	3600.00	7	145520	58.90	3600.00	95	145520	69.65	3600.00	3090
BKZ 510 c	150995	58.82	3600.00	8	150995	59.82	3600.00	162	150995	68.35	3600.00	2490
BKZ 61 c	81151	53.37	3600.00	7	81151	53.87	3600.00	107	81151	61.63	3600.00	2027
BKZ 62 c	85719	53.04	3600.00	7	85719	54.88	3600.00	67	85719	61.06	3600.00	1503
BKZ 63 c	86575	54.71	3600.00	7	86575	56.63	3600.00	154	86575	64.55	3600.00	2057
BKZ 64 c	80752	52.55	3600.00	7	80752	54.50	3600.00	108	80752	61.10	3600.00	1744
BKZ 65 c	81370	56.25	3600.00	6	81370	57.05	3600.00	65	81370	63.36	3600.00	2148
Average		55.32				56.43				63.57		

Table B.10: Comparison among the exact algorithms to the set BKZ for $T = 0.3 \times |M|$.

Instance	$\rho(X^{B\&C})$	PR R Best Dev (%)	PR R Best T (s)	PR R Any Dev (%)	PR R Any T (s)	PR I Best Dev (%)	PR I Best T (s)	PR I Any Dev (%)	PR I Any T (s)
BKZ 41	58447	0.00	11.83	0.04	244.45	**-0.05**	9.76	**-0.05**	281.34
BKZ 42	58448	0.00	22.70	0.00	326.10	**-0.38**	17.74	**-0.38**	341.69
BKZ 43	58777	0.00	16.12	0.00	204.38	**-0.09**	13.76	**-0.09**	237.36
BKZ 44	59553	0.00	14.23	0.00	228.66	0.00	11.29	0.00	271.12
BKZ 45	59176	0.16	12.47	0.17	226.48	**-0.58**	10.92	**-0.58**	262.26
BKZ 46	60106	0.00	35.48	0.02	568.83	0.04	28.00	**-0.11**	600.16
BKZ 47	55839	0.00	19.37	0.00	212.81	0.00	13.38	0.00	220.76
BKZ 48	59524	0.00	12.10	0.00	199.52	0.00	10.96	0.00	237.02
BKZ 49	59170	0.09	18.63	**-0.13**	313.35	0.09	13.13	0.09	346.97
BKZ 410	59555	**-0.27**	25.47	**-0.27**	450.42	0.00	21.26	0.10	491.16
BKZ 51	69422	0.08	23.95	**-0.13**	409.93	0.08	19.70	0.08	478.64
BKZ 52	69670	0.16	36.62	**-0.26**	768.41	0.00	29.43	0.14	601.05
BKZ 53	67493	**-0.16**	27.20	**-0.16**	555.66	0.00	23.80	0.12	600.10
BKZ 54	69199	0.00	56.04	0.00	582.04	**-0.06**	43.90	**-0.06**	600.16
BKZ 55	67127	0.08	24.97	0.11	530.73	**-0.18**	20.86	**-0.18**	600.48
BKZ 56	68944	**-0.19**	33.70	**-0.19**	476.90	0.12	25.99	**-0.19**	518.60
BKZ 57	68477	0.15	23.83	0.15	474.25	0.15	18.99	0.15	600.20
BKZ 58	71281	0.00	31.46	**-0.03**	603.26	0.00	25.74	0.00	600.26
BKZ 59	72181	0.22	23.99	0.22	393.43	**-0.31**	19.08	**-0.31**	414.99
BKZ 510	71039	0.01	89.70	0.01	758.86	0.00	64.09	**-0.06**	600.14
BKZ 61	57985	0.00	17.36	0.00	351.82	0.00	17.38	0.00	501.50
BKZ 62	59691	0.00	20.00	0.00	421.23	0.00	19.30	0.00	583.36
BKZ 63	56383	0.00	21.00	0.00	453.26	0.00	20.89	0.01	600.35
BKZ 64	58339	0.00	13.91	0.00	269.34	0.00	13.98	0.00	382.35
BKZ 65	56454	**-0.22**	21.85	**-0.22**	442.19	0.00	21.61	0.04	600.21
BKZ 41 b	56969	**-0.13**	11.44	**-0.13**	257.96	0.00	10.14	0.00	306.76
BKZ 42 b	58449	0.00	21.54	0.00	275.09	0.00	16.95	**-0.04**	364.10
BKZ 43 b	56346	0.00	13.59	0.00	256.03	0.02	10.91	**-0.15**	284.88
BKZ 44 b	58827	0.00	14.61	0.00	302.91	0.00	12.00	0.00	323.10
BKZ 45 b	57769	**-0.09**	7.63	**-0.09**	142.83	0.03	7.09	0.08	177.19
BKZ 46 b	57481	0.05	20.61	0.05	380.40	0.00	15.18	**-0.14**	410.33
BKZ 47 b	60551	**-0.07**	22.26	**-0.07**	391.53	0.00	16.28	**-0.07**	400.69
BKZ 48 b	59397	0.31	29.88	0.31	535.02	**-0.40**	20.36	**-0.40**	600.26
BKZ 49 b	61517	0.00	57.77	0.00	535.99	**-0.22**	45.27	**-0.22**	577.85
BKZ 410 b	60507	0.24	12.47	0.24	230.11	**-0.35**	10.44	**-0.35**	268.72
BKZ 51 b	70463	0.00	30.96	0.00	457.19	0.00	24.34	0.00	503.19
BKZ 52 b	70386	**-0.21**	77.10	**-0.21**	1018.37	0.08	61.94	0.11	600.64
BKZ 53 b	70100	0.00	30.78	0.04	422.92	**-0.09**	25.53	**-0.09**	474.14
BKZ 54 b	69006	0.27	17.39	0.27	377.33	0.27	13.94	0.27	421.44
BKZ 55 b	69782	**-0.06**	35.71	**-0.06**	757.21	0.00	27.37	0.00	600.33
BKZ 56 b	70959	0.00	34.98	0.00	633.96	0.00	28.63	0.00	600.74
BKZ 57 b	70045	0.07	77.38	0.07	722.34	0.19	60.77	**-0.27**	600.01
BKZ 58 b	70982	0.02	30.22	0.02	512.66	0.03	23.38	**-0.18**	565.51
BKZ 59 b	71001	0.12	44.47	0.16	832.81	0.29	35.65	**-0.31**	600.93
BKZ 510 b	70168	0.12	24.24	0.00	415.26	0.04	19.45	**-0.19**	510.98
BKZ 61 b	55442	0.00	13.57	0.00	282.91	0.00	13.45	0.00	406.95
BKZ 62 b	59867	0.00	24.60	0.00	559.15	0.00	24.41	0.00	601.22
BKZ 63 b	59276	0.00	27.59	0.00	597.72	0.00	27.37	0.00	600.76
BKZ 64 b	61014	0.00	38.94	**-0.02**	908.04	0.00	38.83	0.00	600.01
BKZ 65 b	59575	**-0.02**	19.43	**-0.02**	442.16	0.00	19.69	0.00	600.01
BKZ 41 c	60156	**-0.21**	11.78	**-0.21**	230.44	0.03	9.07	0.03	253.53
BKZ 42 c	57219	0.00	17.02	0.00	340.97	0.16	13.61	**-0.17**	379.27
BKZ 43 c	60069	0.00	16.13	0.00	380.72	**-0.21**	13.79	**-0.21**	442.79
BKZ 44 c	59179	**-0.22**	12.94	**-0.22**	239.84	0.06	9.28	0.06	252.77
BKZ 45 c	58937	0.14	8.37	**-0.24**	158.76	0.14	7.01	0.15	196.21
BKZ 46 c	56852	0.00	9.47	**-0.01**	198.17	0.00	8.36	**-0.01**	291.08
BKZ 47 c	56004	0.00	19.14	**-0.12**	188.16	0.00	14.26	0.00	204.70
BKZ 48 c	56544	**-0.10**	16.94	**-0.10**	368.87	0.03	14.18	0.03	428.88
BKZ 49 c	59412	0.04	52.36	**-0.44**	368.56	0.42	39.21	**-0.44**	409.30
BKZ 410 c	59444	0.03	10.33	0.03	227.98	0.19	9.11	**-0.37**	264.63
BKZ 51 c	69854	**-0.29**	26.89	**-0.29**	462.45	0.14	19.50	0.14	506.59
BKZ 52 c	67593	0.00	39.83	**-0.21**	499.96	0.07	33.34	0.12	566.31
BKZ 53 c	71241	0.17	66.67	0.17	882.42	**0.61**	49.29	**-0.61**	600.90
BKZ 54 c	69988	0.00	34.56	0.32	654.67	0.05	30.76	0.47	600.33
BKZ 55 c	67983	**-0.15**	21.89	**-0.15**	412.26	0.05	17.42	0.08	497.46
BKZ 56 c	70661	0.10	46.37	**-0.40**	859.46	0.02	39.03	0.06	600.53
BKZ 57 c	71148	0.12	174.38	0.12	1130.96	0.26	131.12	**-0.53**	600.02
BKZ 58 c	69949	0.00	16.19	**-0.19**	311.45	0.00	12.80	0.00	352.14
BKZ 59 c	68918	0.17	50.93	0.21	736.28	0.30	40.00	**-0.34**	600.23
BKZ 510 c	73105	0.22	28.61	0.22	640.63	0.21	25.04	0.21	600.08
BKZ 61 c	58522	0.00	9.54	0.00	188.91	0.00	9.52	0.00	271.21
BKZ 62 c	56937	0.00	14.75	**-0.02**	348.09	0.00	15.44	0.00	504.22
BKZ 63 c	59309	0.00	24.61	0.00	560.82	0.00	24.15	0.00	600.01
BKZ 64 c	57540	0.01	14.19	0.01	308.67	0.01	14.26	0.01	442.45
BKZ 65 c	53612	0.00	19.47	0.00	357.16	0.00	19.38	0.00	521.44
Average		0.07	28.75	0.10	450.28	0.09	23.51	**-0.13**	460.80

Table B.11: Comparison among the Path Relinking strategies to the set BKZ for $T = 0.1 \times |M|$.

Instance	$\rho(X^{\text{B\&C}})$	PR R Best Dev (%)	PR R Best T (s)	PR R Any Dev (%)	PR R Any T (s)	PR I Best Dev (%)	PR I Best T (s)	PR I Any Dev (%)	PR I Any T (s)
BKZ 41	81910	0.00	50.20	0.00	3600.00	0.00	53.35	0.00	3600.00
BKZ 42	83679	0.06	87.53	0.06	3600.00	0.06	89.17	0.06	3600.00
BKZ 43	80563	0.25	38.23	0.25	3600.00	0.25	39.21	0.25	3600.00
BKZ 44	82706	0.00	64.79	0.00	3600.00	0.00	64.23	0.00	3600.00
BKZ 45	84004	0.00	71.66	0.00	3600.00	0.00	71.94	0.00	3600.00
BKZ 46	82552	0.00	66.61	0.00	3600.00	0.00	70.59	0.00	3600.00
BKZ 47	80877	0.01	65.01	0.01	3600.00	0.01	65.38	0.01	3600.00
BKZ 48	83906	0.00	76.07	0.00	3600.00	0.00	76.01	0.00	3600.00
BKZ 49	81427	0.00	82.22	0.00	3600.00	0.00	86.15	0.00	3600.00
BKZ 410	77683	0.08	57.19	0.08	3600.00	0.08	55.10	0.07	3600.00
BKZ 51	116839	0.00	117.91	0.00	3600.00	0.00	118.81	0.00	3600.00
BKZ 52	113746	0.00	179.58	0.00	3600.00	0.00	181.94	0.00	3600.00
BKZ 53	112937	0.00	144.94	0.00	3600.00	0.00	148.16	0.00	3600.00
BKZ 54	117507	0.13	227.23	0.22	3600.00	0.22	231.17	0.05	3600.00
BKZ 55	115021	0.07	137.98	0.10	3600.00	0.10	140.06	0.07	3600.00
BKZ 56	110857	0.00	87.05	0.00	3600.00	0.00	86.84	0.00	3600.00
BKZ 57	115320	0.01	192.83	0.00	3600.00	0.00	192.22	0.00	3600.00
BKZ 58	115598	0.00	130.49	0.00	3600.00	0.00	133.10	0.00	3600.00
BKZ 59	115578	0.00	91.98	0.00	3600.00	0.00	90.63	0.00	3600.00
BKZ 510	115851	0.13	142.26	0.13	3600.00	0.13	144.42	0.13	3600.00
BKZ 61	79185	0.00	110.87	0.00	3600.00	0.00	110.54	0.00	3600.00
BKZ 62	81344	0.00	98.29	0.00	3600.00	0.00	98.70	0.00	3600.00
BKZ 63	79315	0.00	101.37	0.00	3600.00	0.00	100.66	0.00	3600.00
BKZ 64	83163	0.00	150.70	0.00	3600.00	0.00	151.15	0.00	3600.00
BKZ 65	74905	0.00	104.67	0.00	3600.00	0.00	105.55	0.00	3600.00
BKZ 41 b	80312	0.00	56.64	0.00	3600.00	0.00	57.77	0.00	3600.00
BKZ 42 b	80717	0.00	65.17	0.00	3600.00	0.00	67.80	0.00	3600.00
BKZ 43 b	83095	0.16	74.36	0.16	3600.00	0.16	77.45	0.16	3600.00
BKZ 44 b	83034	0.00	65.73	0.00	3600.00	0.00	66.15	0.00	3600.00
BKZ 45 b	79188	0.00	52.14	0.00	3600.00	0.00	53.14	0.00	3600.00
BKZ 46 b	74510	0.02	49.46	0.02	3600.00	0.00	49.58	0.00	3600.00
BKZ 47 b	79918	0.00	50.61	0.00	3600.00	0.00	49.61	0.00	3600.00
BKZ 48 b	85962	0.00	55.84	0.00	3600.00	0.00	57.28	0.00	3600.00
BKZ 49 b	86290	0.04	91.71	0.04	3600.00	0.04	93.29	0.04	3600.00
BKZ 410 b	80130	0.00	49.87	0.00	3600.00	0.00	50.04	0.00	3600.00
BKZ 51 b	113566	0.02	119.69	0.01	3600.00	0.01	128.59	0.01	3600.00
BKZ 52 b	114953	0.04	170.57	0.08	3600.00	0.08	169.69	0.00	3600.00
BKZ 53 b	114305	0.03	136.69	0.06	3600.00	0.06	136.48	0.03	3600.00
BKZ 54 b	112282	0.14	153.28	0.13	3600.00	0.13	156.38	0.11	3600.00
BKZ 55 b	118668	0.00	138.99	0.00	3600.00	0.00	143.24	0.00	3600.00
BKZ 56 b	118053	0.08	150.31	0.08	3600.00	0.08	149.48	0.08	3600.00
BKZ 57 b	113942	0.00	116.98	0.00	3600.00	0.00	117.50	0.00	3600.00
BKZ 58 b	120248	0.03	138.84	0.01	3600.00	0.01	139.55	0.01	3600.00
BKZ 59 b	115389	0.00	112.75	0.00	3600.00	0.00	113.41	0.00	3600.00
BKZ 510 b	116495	0.00	126.41	0.00	3600.00	0.00	127.47	0.00	3600.00
BKZ 61 b	78572	0.00	131.01	0.00	3600.00	0.00	131.00	0.00	3600.00
BKZ 62 b	78369	0.00	69.36	0.00	3600.00	0.00	71.60	0.00	3600.00
BKZ 63 b	78510	0.00	89.98	0.00	3600.00	0.00	90.87	0.00	3600.00
BKZ 64 b	79162	7.18	120.99	7.18	3600.00	7.18	121.09	7.18	3600.00
BKZ 65 b	75407	13.80	161.99	13.80	3600.00	13.80	162.93	13.80	3600.00
BKZ 41 c	83162	0.00	71.58	0.00	3600.00	0.00	70.65	0.00	3600.00
BKZ 42 c	81866	0.00	74.08	0.00	3600.00	0.00	73.90	0.00	3600.00
BKZ 43 c	83110	0.00	52.98	0.00	3600.00	0.00	55.15	0.00	3600.00
BKZ 44 c	82259	0.15	49.84	0.15	3600.00	0.10	51.04	0.02	3600.00
BKZ 45 c	82734	0.00	64.44	0.00	3600.00	0.00	65.36	0.00	3600.00
BKZ 46 c	75550	0.00	60.24	0.00	3600.00	0.00	62.25	0.00	3600.00
BKZ 47 c	79937	0.00	52.89	0.03	3600.00	0.03	53.61	0.00	3600.00
BKZ 48 c	79368	0.10	51.00	0.10	3600.00	0.10	50.30	0.02	3600.00
BKZ 49 c	84649	0.02	63.32	0.00	3600.00	0.00	64.31	0.00	3600.00
BKZ 410 c	81534	0.22	78.96	0.22	3600.00	0.22	80.60	0.22	3600.00
BKZ 51 c	116493	0.00	148.80	0.00	3600.00	0.00	152.02	0.00	3600.00
BKZ 52 c	111826	0.00	162.88	0.00	3600.00	0.00	164.58	0.00	3600.00
BKZ 53 c	117257	0.00	129.98	0.00	3600.00	0.00	131.13	0.00	3600.00
BKZ 54 c	117653	0.01	124.33	0.01	3600.00	0.01	124.67	0.00	3600.00
BKZ 55 c	110144	0.01	91.61	0.01	3600.00	0.01	90.60	0.00	3600.00
BKZ 56 c	113897	0.00	104.26	0.00	3600.00	0.00	105.50	0.00	3600.00
BKZ 57 c	116775	0.03	135.26	0.03	3600.00	0.03	140.74	0.03	3600.00
BKZ 58 c	114538	0.00	114.70	0.00	3600.00	0.00	113.62	0.00	3600.00
BKZ 59 c	114481	0.00	148.87	0.00	3600.00	0.00	145.43	0.00	3600.00
BKZ 510 c	120370	0.00	150.40	0.00	3600.00	0.00	151.28	0.00	3600.00
BKZ 61 c	82234	0.00	113.36	0.00	3600.00	0.00	113.20	0.00	3600.00
BKZ 62 c	80523	0.02	101.76	0.01	3600.00	0.01	101.85	0.01	3600.00
BKZ 63 c	83558	0.02	132.48	0.02	3600.00	0.02	133.11	0.00	3600.00
BKZ 64 c	79162	0.00	86.28	0.00	3600.00	0.00	86.30	0.00	3600.00
BKZ 65 c	75407	0.07	115.48	0.07	3600.00	0.07	115.59	0.07	3600.00
Average		0.25	102.76	0.25	3600.00	0.25	103.78	0.26	3600.00

Table B.12: Comparison among the Path Relinking strategies to the set BKZ for $T = 0.2 \times |M|$.

Instance	$\rho(X^{B\&C})$	PR R Best		PR R Any		PR I Best		PR I Any	
		Dev (%)	T (s)	Dev (%)	T (s)	Dev (%)	T (s)	Dev (%)	T (s)
BKZ 41	84057	0.00	128.38	0.00	3600.00	0.00	127.88	0.00	3600.00
BKZ 42	86949	0.00	156.05	0.00	3600.00	0.00	156.84	0.00	3600.00
BKZ 43	81116	0.00	108.31	0.00	3600.00	0.00	109.17	0.00	3600.00
BKZ 44	84919	0.00	103.08	0.00	3600.00	0.00	104.01	0.00	3600.00
BKZ 45	88603	0.00	159.90	0.00	3600.00	0.00	156.94	0.00	3600.00
BKZ 46	85543	0.00	142.55	0.00	3600.00	0.00	145.33	0.00	3600.00
BKZ 47	83015	0.00	133.53	0.00	3600.00	0.00	132.61	0.00	3600.00
BKZ 48	86638	0.00	148.08	0.00	3600.00	0.00	147.97	0.00	3600.00
BKZ 49	80300	0.00	121.33	0.00	3600.00	0.00	118.50	0.00	3600.00
BKZ 410	78350	0.00	141.32	0.00	3600.00	0.00	142.51	0.00	3600.00
BKZ 51	146862	0.00	411.06	0.00	3600.00	0.00	409.50	0.00	3600.00
BKZ 52	142741	0.00	335.64	0.00	3600.00	0.00	336.04	0.00	3600.00
BKZ 53	145054	0.00	347.56	0.00	3600.00	0.00	346.39	0.00	3600.00
BKZ 54	147638	0.00	495.37	0.00	3600.00	0.00	486.35	0.00	3600.00
BKZ 55	146392	0.01	330.06	0.01	3600.00	0.01	325.59	0.01	3600.00
BKZ 56	140342	0.00	389.45	0.00	3600.00	0.00	389.83	0.00	3600.00
BKZ 57	148061	0.00	470.88	0.00	3600.00	0.00	476.07	0.00	3600.00
BKZ 58	142878	0.00	371.71	0.00	3600.00	0.00	364.80	0.00	3600.00
BKZ 59	140573	0.00	314.36	0.00	3600.00	0.00	315.02	0.00	3600.00
BKZ 510	144963	0.00	557.09	0.00	3600.00	0.00	547.95	0.00	3600.00
BKZ 61	80727	0.00	173.10	0.00	3600.00	0.00	172.94	0.00	3600.00
BKZ 62	83328	0.00	229.71	0.00	3600.00	0.00	226.90	0.00	3600.00
BKZ 63	83817	0.00	190.17	0.00	3600.00	0.00	190.49	0.00	3600.00
BKZ 64	84911	0.00	249.88	0.00	3600.00	0.00	249.13	0.00	3600.00
BKZ 65	76058	0.00	255.83	0.00	3600.00	0.00	258.77	0.00	3600.00
BKZ 41 b	83109	0.00	148.80	0.00	3600.00	0.00	149.05	0.00	3600.00
BKZ 42 b	83346	0.00	137.09	0.00	3600.00	0.00	136.93	0.00	3600.00
BKZ 43 b	86101	0.00	166.95	0.00	3600.00	0.00	161.08	0.00	3600.00
BKZ 44 b	84309	0.00	131.11	0.00	3600.00	0.00	132.04	0.00	3600.00
BKZ 45 b	82577	0.00	104.50	0.00	3600.00	0.00	102.58	0.00	3600.00
BKZ 46 b	75800	0.00	120.19	0.00	3600.00	0.00	120.12	0.00	3600.00
BKZ 47 b	79865	0.00	96.72	0.00	3600.00	0.00	97.04	0.00	3600.00
BKZ 48 b	91065	0.00	133.20	0.00	3600.00	0.00	132.27	0.00	3600.00
BKZ 49 b	88552	0.00	126.84	0.00	3600.00	0.00	127.28	0.00	3600.00
BKZ 410 b	79189	0.00	108.10	0.00	3600.00	0.00	107.98	0.00	3600.00
BKZ 51 b	142540	0.00	319.50	0.00	3600.00	0.00	318.16	0.00	3600.00
BKZ 52 b	141960	0.00	419.54	0.00	3600.00	0.00	421.12	0.00	3600.00
BKZ 53 b	139088	0.00	384.11	0.00	3600.00	0.00	386.31	0.00	3600.00
BKZ 54 b	141161	0.00	357.85	0.00	3600.00	0.00	358.61	0.00	3600.00
BKZ 55 b	149953	0.00	441.97	0.00	3600.00	0.00	447.29	0.00	3600.00
BKZ 56 b	146290	0.00	452.42	0.00	3600.00	0.00	450.56	0.00	3600.00
BKZ 57 b	143834	0.00	437.71	0.00	3600.00	0.00	438.46	0.00	3600.00
BKZ 58 b	150488	0.04	334.21	0.04	3600.00	0.04	334.78	0.04	3600.00
BKZ 59 b	144310	0.00	476.47	0.00	3600.00	0.00	475.75	0.00	3600.00
BKZ 510 b	145583	0.00	372.54	0.00	3600.00	0.00	370.73	0.00	3600.00
BKZ 61 b	79698	0.00	235.66	0.00	3600.00	0.00	236.87	0.00	3600.00
BKZ 62 b	77616	0.00	198.39	0.00	3600.00	0.00	197.61	0.00	3600.00
BKZ 63 b	82638	0.00	227.70	0.00	3600.00	0.00	227.16	0.00	3600.00
BKZ 64 b	86076	0.00	213.91	0.00	3600.00	0.00	215.92	0.00	3600.00
BKZ 65 b	88496	0.00	212.12	0.00	3600.00	0.00	211.71	0.00	3600.00
BKZ 41 c	81126	0.00	101.18	0.00	3600.00	0.00	101.07	0.00	3600.00
BKZ 42 c	88322	0.00	145.71	0.00	3600.00	0.00	145.23	0.00	3600.00
BKZ 43 c	86852	0.00	104.76	0.00	3600.00	0.00	104.19	0.00	3600.00
BKZ 44 c	86673	0.00	163.94	0.00	3600.00	0.00	164.65	0.00	3600.00
BKZ 45 c	86569	0.00	104.93	0.00	3600.00	0.00	104.21	0.00	3600.00
BKZ 46 c	77894	0.00	104.61	0.00	3600.00	0.00	104.56	0.00	3600.00
BKZ 47 c	82091	0.00	165.70	0.00	3600.00	0.00	162.21	0.00	3600.00
BKZ 48 c	83589	0.00	170.44	0.00	3600.00	0.00	172.05	0.00	3600.00
BKZ 49 c	88519	0.00	25.59	0.00	3600.00	0.00	105.57	0.00	3600.00
BKZ 410 c	82546	0.00	24.04	0.00	3600.00	0.00	112.63	0.00	3600.00
BKZ 51 c	146996	0.00	80.72	0.00	3600.00	0.00	341.47	0.00	3600.00
BKZ 52 c	140764	0.01	81.12	0.01	3600.00	0.01	389.91	0.01	3600.00
BKZ 53 c	145415	0.00	80.67	0.00	3600.00	0.00	292.36	0.00	3600.00
BKZ 54 c	148148	0.00	89.51	0.00	3600.00	0.00	454.99	0.00	3600.00
BKZ 55 c	139314	0.00	82.69	0.00	3600.00	0.00	341.54	0.00	3600.00
BKZ 56 c	142511	0.00	89.00	0.00	3600.00	0.00	390.58	0.00	3600.00
BKZ 57 c	143687	0.00	87.82	0.00	3600.00	0.00	415.72	0.00	3600.00
BKZ 58 c	143999	0.00	86.27	0.00	3600.00	0.00	390.74	0.00	3600.00
BKZ 59 c	145520	0.01	84.69	0.01	3600.00	0.01	430.13	0.01	3600.00
BKZ 510 c	150995	0.00	85.30	0.00	3600.00	0.00	367.98	0.00	3600.00
BKZ 61 c	81151	0.00	36.23	0.00	3600.00	0.00	132.00	0.00	3600.00
BKZ 62 c	85719	0.00	31.09	0.00	3600.00	0.00	215.92	0.00	3600.00
BKZ 63 c	86575	0.00	30.93	0.00	3600.00	0.00	244.46	0.00	3600.00
BKZ 64 c	80752	0.00	32.69	0.00	3600.00	0.00	214.52	0.00	3600.00
BKZ 65 c	81370	0.00	36.61	0.00	3600.00	0.00	300.93	0.00	3600.00
Average		0.00	253.26	0.00	3600.00	0.00	199.31	0.00	3600.00

Table B.13: Comparison among the Path Relinking strategies to the set BKZ for $T = 0.3 \times |M|$.

		AMU		SBA		PR I Any		LPH		PM	
Instance	$\rho(X^{\text{B\&C}})$	Dev (%)	T (s)	Dev (%)	T(s)	Dev (%)	T (s)	Dev (%)	T (s)	Dev (%)	T (s)
BKZ 41	58447	0.00	2.25	0.00	2.26	0.05	281.34	0.00	3600.00	**-0.23**	16.21
BKZ 42	58448	0.00	2.18	0.00	2.15	0.38	341.69	**-1.31**	286.98	0.95	15.05
BKZ 43	58777	0.00	2.19	0.00	2.15	0.09	237.36	**-0.54**	644.12	1.68	13.87
BKZ 44	59553	**0.00**	2.21	**0.00**	2.19	**0.00**	271.12	1.87	3600.00	1.34	18.62
BKZ 45	59176	0.00	2.05	0.00	2.10	0.58	262.26	**-0.69**	542.33	0.58	17.5
BKZ 46	60106	0.00	2.15	0.00	2.17	**-0.11**	600.16	1.57	3600.00	1.83	16.12
BKZ 47	55839	**0.00**	2.11	**0.00**	2.10	**0.00**	220.76	0.65	3600.00	1.12	14.78
BKZ 48	59524	**0.00**	2.21	**0.00**	2.16	**0.00**	237.02	0.19	3600.00	0.08	13.93
BKZ 49	59170	0.00	2.13	0.00	2.10	**-0.09**	346.97	0.14	3600.00	1.57	14.71
BKZ 410	59555	0.00	2.14	0.00	2.13	**-0.10**	491.16	0.48	3600.00	1.18	14.2
BKZ 51	69422	0.00	4.09	0.00	4.13	0.08	478.64	**-0.57**	3600.00	0.40	45.7
BKZ 52	69670	0.00	4.06	0.00	4.07	0.14	601.05	**-0.31**	3600.00	0.80	52.59
BKZ 53	67493	0.00	4.06	0.00	4.09	**-0.12**	600.10	0.35	3600.00	0.81	54.54
BKZ 54	69199	0.00	4.10	0.00	4.09	0.06	600.16	**-1.14**	3600.00	2.25	52.86
BKZ 55	67127	0.00	3.88	0.00	3.95	**-0.18**	600.48	0.14	3600.00	0.64	44.99
BKZ 56	68944	0.00	3.90	0.00	3.91	**-0.19**	518.60	0.12	3600.00	0.48	51.48
BKZ 57	68477	0.00	4.00	0.00	4.02	**-0.15**	600.20	0.47	3600.00	0.44	42.87
BKZ 58	71281	**0.00**	3.90	**0.00**	3.93	**0.00**	600.26	0.27	3600.00	0.42	47.54
BKZ 59	72181	0.00	3.93	0.00	3.90	0.31	414.99	**-1.21**	3600.00	0.30	52.27
BKZ 510	71039	0.00	3.94	0.00	3.91	0.06	600.14	**-1.04**	3600.00	0.59	52.2
BKZ 61	57985	**0.00**	2.77	**0.00**	2.79	**0.00**	501.50	2.24	18.31	1.01	20.98
BKZ 62	59691	**0.00**	2.77	**0.00**	2.79	**0.00**	583.36	3.03	10.35	1.06	21.25
BKZ 63	56383	0.00	2.81	0.00	2.73	**-0.01**	600.35	4.25	19.75	1.59	22.62
BKZ 64	58339	**0.00**	2.83	**0.00**	2.78	**0.00**	382.35	3.05	28.64	1.15	25.45
BKZ 65	56454	0.00	2.73	0.00	2.81	**-0.04**	600.21	0.87	148.81	0.90	26.49
BKZ 41 b	56969	**0.00**	2.15	**0.00**	2.16	**0.00**	306.76	0.86	3600.00	1.16	15.42
BKZ 42 b	58449	0.00	2.14	0.00	2.17	**-0.04**	364.10	0.09	3600.00	1.35	17.02
BKZ 43 b	56346	0.00	2.17	0.00	2.09	**-0.15**	284.88	1.27	3600.00	1.69	14.56
BKZ 44 b	58827	**0.00**	2.19	**0.00**	2.21	**0.00**	323.10	0.03	3600.00	2.76	16.6
BKZ 45 b	57769	0.00	2.13	0.00	2.16	**-0.08**	177.19	1.02	3600.00	2.29	15.95
BKZ 46 b	57481	0.00	2.14	0.00	2.16	**-0.14**	410.33	0.33	3600.00	1.49	15.93
BKZ 47 b	60551	0.00	2.15	0.00	2.16	0.07	400.69	**-0.26**	3600.00	1.27	14.29
BKZ 48 b	59397	0.00	2.12	0.00	2.13	**-0.40**	600.26	0.40	3600.00	0.75	14.48
BKZ 49 b	61517	0.00	2.18	0.00	2.16	0.22	577.85	**-2.51**	3600.00	0.88	15.35
BKZ 410 b	60507	0.00	2.14	0.00	2.14	0.35	268.72	**-0.96**	3600.00	0.13	14.56
BKZ 51 b	70463	**0.00**	3.92	**0.00**	4.01	**0.00**	503.19	0.89	3600.00	0.84	44.71
BKZ 52 b	70386	0.00	4.01	0.00	4.02	0.11	600.64	**-1.24**	3600.00	2.01	51.63
BKZ 53 b	70100	0.00	3.97	0.00	3.90	0.09	474.14	**-0.86**	3600.00	1.09	49.94
BKZ 54 b	69006	0.00	3.96	0.00	4.00	**-0.26**	421.44	0.28	3600.00	0.15	50.09
BKZ 55 b	69782	0.00	3.87	0.00	3.92	0.00	630.33	**-0.30**	3600.00	1.24	47.92
BKZ 56 b	70959	0.00	3.95	0.00	4.00	0.00	570.74	**-0.64**	3600.00	0.55	52.33
BKZ 57 b	70045	0.00	4.06	0.00	4.05	**-0.27**	600.01	0.18	3600.00	0.06	57.98
BKZ 58 b	70982	0.00	3.91	0.00	3.90	0.18	565.51	**-0.38**	3600.00	0.85	45.46
BKZ 59 b	71001	0.00	3.96	0.00	3.93	0.31	812.93	**-0.74**	3600.00	0.81	50.31
BKZ 510 b	70168	0.00	3.92	0.00	3.90	**-0.19**	510.98	0.36	3600.00	0.74	52.81
BKZ 61 b	55442	**0.00**	2.77	**0.00**	2.78	**0.00**	406.95	2.48	49.75	1.07	19.14
BKZ 62 b	59867	**0.00**	2.80	**0.00**	2.77	**0.00**	601.22	1.53	174.1	1.06	20.9
BKZ 63 b	59276	**0.00**	2.79	**0.00**	2.81	**0.00**	600.76	2.93	253.93	1.21	24.11
BKZ 64 b	61014	**0.00**	2.81	0.00	2.78	**0.00**	712.01	2.05	32.73	1.65	20.28
BKZ 65 b	59575	**0.00**	2.73	0.00	2.79	**0.00**	600.46	2.48	14.1	1.41	22.87
BKZ 41 c	60156	0.00	2.08	0.00	2.12	0.03	253.53	**-0.52**	3600.00	0.47	15.05
BKZ 42 c	57219	0.00	2.09	0.00	2.08	**-0.17**	379.27	0.07	3600.00	1.02	15.11
BKZ 43 c	60069	0.00	2.18	0.00	2.18	**-0.21**	442.79	0.05	3600.00	0.80	12.9
BKZ 44 c	59179	0.00	2.13	0.00	2.11	**-0.06**	252.77	1.26	3600.00	0.03	14.74
BKZ 45 c	58937	0.00	2.11	0.00	2.15	**-0.15**	196.21	0.59	3600.00	0.18	15.04
BKZ 46 c	56852	0.00	2.09	0.00	2.10	**-0.01**	291.08	0.33	3600.00	0.13	13.78
BKZ 47 c	56004	0.00	2.17	0.00	2.18	0.00	204.70	0.17	3600.00	1.24	14.39
BKZ 48 c	56544	0.00	2.17	0.00	2.16	0.03	428.88	**-1.22**	3600.00	1.10	13.82
BKZ 49 c	59412	0.00	2.13	0.00	2.19	0.44	409.30	**-0.98**	3600.00	0.28	14.87
BKZ 410 c	59444	0.00	2.22	0.00	2.23	0.37	264.63	**-0.47**	437.6	0.03	15.19
BKZ 51 c	69854	0.00	4.05	0.00	4.08	0.14	506.59	**0.39**	3600.00	0.21	49.4
BKZ 52 c	67593	0.00	4.08	0.00	4.08	0.12	566.31	**-0.20**	3600.00	1.81	48.68
BKZ 53 c	71241	0.00	3.92	0.00	3.97	**-0.61**	600.90	0.48	3600.00	0.65	47.33
BKZ 54 c	69988	0.00	4.09	0.00	4.11	0.47	600.33	**-0.63**	3600.00	1.53	53.1
BKZ 55 c	67983	0.00	4.25	0.00	4.25	**-0.08**	497.46	**0.08**	3600.00	0.17	58.74
BKZ 56 c	70661	0.00	3.89	0.00	3.90	0.06	600.53	**-0.35**	3600.00	0.97	45.87
BKZ 57 c	71148	0.00	4.03	0.00	4.03	**-0.53**	600.02	0.39	3600.00	0.44	54.54
BKZ 58 c	69949	**0.00**	3.96	**0.00**	3.97	**0.00**	352.14	0.21	3600.00	0.81	49.19
BKZ 59 c	68918	0.00	3.97	0.00	3.90	0.34	930.23	**-0.37**	3600.00	1.02	54.39
BKZ 510 c	73105	0.00	4.02	0.00	3.98	0.21	1600.08	**-0.36**	3600.00	0.43	44.52
BKZ 61 c	58522	**0.00**	2.78	**0.00**	2.82	**0.00**	271.21	2.62	26.78	0.68	27.32
BKZ 62 c	56937	**0.00**	2.81	**0.00**	2.78	**0.00**	504.22	3.01	24.84	1.71	21.16
BKZ 63 c	59309	**0.00**	2.82	**0.00**	2.77	**0.00**	600.01	2.77	68.31	1.49	19.11
BKZ 64 c	57540	0.00	2.75	0.00	2.81	**-0.01**	442.45	3.63	11.47	1.05	18.8
BKZ 65 c	53612	**0.00**	2.81	**0.00**	2.82	**0.00**	521.44	2.07	20.85	1.43	21.38
Average		0.00	3.01	0.00	3.02	**-0.13**	460.80	0.44	2725.52	0.91	30.56

Table B.14: Comparison among the proposed heuristics to the set BKZ for $T = 0.1 \times |M|$.

Instance	$\rho(X^{B\&C})$	AMU		SBA		PR R Any		LPH		PM	
		Dev (%)	T (s)	Dev (%)	T(s)	Dev (%)	T (s)	Dev (%)	T (s)	Dev (%)	T (s)
BKZ 41	81910	0.00	0.07	0.00	2.67	0.00	3600.00	13.62	13.88	2.57	22.19
BKZ 42	83679	0.00	0.07	0.06	2.64	0.06	3600.00	13.81	9.11	1.64	21.10
BKZ 43	80563	0.00	0.07	0.00	2.64	0.25	3600.00	13.83	6.89	1.99	21.90
BKZ 44	82706	0.00	0.07	0.00	2.63	0.00	3600.00	17.33	10.94	1.58	20.64
BKZ 45	84004	0.00	0.07	0.00	2.63	0.00	3600.00	20.33	15.92	2.41	19.81
BKZ 46	82552	0.00	0.06	0.00	2.66	0.00	3600.00	13.01	11.51	2.35	19.72
BKZ 47	80877	0.00	0.07	0.01	2.62	0.01	3600.00	19.21	13.70	3.05	18.63
BKZ 48	83906	0.00	0.06	0.00	2.60	0.00	3600.00	18.01	10.08	2.55	18.16
BKZ 49	81427	0.00	0.08	0.00	2.62	0.00	3600.00	9.22	13.57	2.35	21.63
BKZ 410	77683	0.00	0.08	0.07	2.61	0.08	3600.00	12.08	9.74	2.18	18.69
BKZ 51	116839	0.00	0.13	0.00	4.84	0.00	3600.00	31.04	161.70	1.73	66.53
BKZ 52	113746	0.00	0.13	0.00	4.85	0.00	3600.00	33.07	343.42	0.97	69.18
BKZ 53	112937	0.00	0.14	0.00	4.92	0.00	3600.00	33.41	58.78	1.90	68.01
BKZ 54	117507	0.00	0.13	0.05	4.88	0.22	3600.00	34.33	84.94	1.15	65.64
BKZ 55	115021	0.00	0.12	0.07	4.83	0.10	3600.00	35.31	409.44	1.19	63.72
BKZ 56	110857	0.00	0.13	0.00	4.88	0.00	3600.00	32.95	70.65	0.77	70.55
BKZ 57	115320	0.00	0.13	0.00	4.89	0.00	3600.00	35.16	108.38	1.06	68.62
BKZ 58	115598	0.00	0.12	0.00	4.89	0.00	3600.00	30.11	144.28	1.02	73.23
BKZ 59	115578	0.00	0.13	0.00	4.86	0.00	3600.00	27.81	45.86	1.21	70.25
BKZ 510	115851	0.00	0.13	0.13	4.93	0.13	3600.00	30.65	212.95	0.65	65.30
BKZ 61	79185	0.00	0.09	0.00	3.33	0.00	3600.00	14.51	5.20	1.55	21.63
BKZ 62	81344	0.00	0.09	0.00	3.34	0.00	3600.00	13.06	3.31	2.15	28.32
BKZ 63	79315	0.00	0.09	0.00	3.36	0.00	3600.00	15.29	6.51	3.28	27.77
BKZ 64	83163	0.00	0.09	0.00	3.33	0.00	3600.00	17.31	11.64	2.63	27.38
BKZ 65	74905	0.00	0.09	0.00	3.36	0.00	3600.00	19.80	9.67	3.28	30.32
BKZ 41 b	80312	0.00	0.08	0.00	2.62	0.00	3600.00	15.53	10.95	2.40	21.04
BKZ 42 b	80717	0.00	0.08	0.00	2.61	0.00	3600.00	13.66	14.07	2.32	20.36
BKZ 43 b	83095	0.00	0.08	0.16	2.62	0.16	3600.00	19.32	15.42	2.44	19.78
BKZ 44 b	83034	0.00	0.07	0.00	2.63	0.00	3600.00	11.75	14.05	1.46	20.07
BKZ 45 b	79188	0.00	0.07	0.00	2.65	0.00	3600.00	18.32	11.22	1.91	22.62
BKZ 46 b	74510	0.00	0.08	0.00	2.66	0.02	3600.00	15.98	7.77	3.58	22.17
BKZ 47 b	79918	0.00	0.06	0.00	2.63	0.00	3600.00	11.00	7.93	2.69	19.06
BKZ 48 b	85962	0.00	0.08	0.00	2.63	0.00	3600.00	14.94	9.10	2.07	18.84
BKZ 49 b	86290	0.00	0.07	0.04	2.59	0.04	3600.00	11.96	11.64	1.84	21.06
BKZ 410 b	80130	0.00	0.08	0.00	2.63	0.00	3600.00	12.11	3.86	2.22	18.74
BKZ 51 b	113566	0.00	0.13	0.01	4.89	0.01	3600.00	32.68	35.91	0.73	60.42
BKZ 52 b	114953	0.00	0.13	0.00	4.87	0.08	3600.00	28.89	74.61	1.26	62.90
BKZ 53 b	114305	0.00	0.13	0.03	4.92	0.06	3600.00	27.47	361.94	1.35	62.99
BKZ 54 b	112282	0.00	0.13	0.11	4.87	0.13	3600.00	31.06	76.12	1.00	72.27
BKZ 55 b	118668	0.00	0.13	0.00	4.85	0.00	3600.00	32.86	416.87	1.08	64.85
BKZ 56 b	118053	0.00	0.14	0.08	4.84	0.08	3600.00	30.20	54.81	1.31	59.52
BKZ 57 b	113942	0.00	0.14	0.00	4.94	0.00	3600.00	30.39	899.69	0.84	75.81
BKZ 58 b	120248	0.00	0.14	0.01	4.88	0.01	3600.00	32.31	196.99	0.96	59.39
BKZ 59 b	115389	0.00	0.13	0.00	4.89	0.00	3600.00	31.79	131.35	1.21	79.06
BKZ 510 b	116495	0.00	0.13	0.00	4.85	0.00	3600.00	29.96	545.91	0.85	65.67
BKZ 61 b	78572	0.00	0.09	0.01	3.30	0.00	3600.00	13.43	5.53	3.27	23.77
BKZ 62 b	78369	0.00	0.08	0.00	3.36	0.00	3600.00	10.14	3.93	2.61	30.27
BKZ 63 b	78510	0.00	0.09	0.00	3.32	0.00	3600.00	14.80	12.23	1.92	29.03
BKZ 64 b	79162	7.18	0.09	7.18	3.36	7.18	3600.00	20.43	5.59	9.45	24.91
BKZ 65 b	75407	13.80	0.09	13.80	3.32	13.80	3600.00	35.10	9.86	16.49	26.27
BKZ 41 c	83162	0.00	0.08	0.00	2.63	0.00	3600.00	7.23	10.43	2.09	20.28
BKZ 42 c	81866	0.00	0.07	0.00	2.62	0.00	3600.00	16.73	16.82	4.96	19.71
BKZ 43 c	83110	0.00	0.08	0.00	2.62	0.00	3600.00	12.58	26.40	2.46	21.39
BKZ 44 c	82259	0.00	0.07	0.00	2.64	0.15	3600.00	15.20	9.06	2.41	20.23
BKZ 45 c	82734	0.00	0.07	0.00	2.62	0.00	3600.00	16.30	10.27	2.13	21.91
BKZ 46 c	75550	0.00	0.07	0.00	2.65	0.00	3600.00	19.33	9.51	2.57	17.64
BKZ 47 c	79937	0.00	0.07	0.00	2.63	0.03	3600.00	15.47	7.43	3.11	18.22
BKZ 48 c	79368	0.00	0.07	0.02	2.62	0.10	3600.00	20.15	12.50	2.92	20.65
BKZ 49 c	84649	0.00	0.07	0.00	2.57	0.00	3600.00	14.97	8.05	1.64	21.47
BKZ 410 c	81534	0.00	0.07	0.22	2.65	0.22	3600.00	14.00	6.01	1.63	20.84
BKZ 51 c	116493	0.00	0.12	0.00	4.88	0.00	3600.00	32.96	208.71	1.45	64.94
BKZ 52 c	111826	0.00	0.13	0.00	4.81	0.00	3600.00	33.71	100.97	1.57	59.91
BKZ 53 c	117257	0.00	0.12	0.00	4.86	0.00	3600.00	31.03	85.37	0.79	77.08
BKZ 54 c	117653	0.00	0.13	0.00	4.87	0.01	3600.00	30.28	899.55	1.19	70.96
BKZ 55 c	110144	0.00	0.13	0.00	4.86	0.01	3600.00	33.44	112.77	1.18	82.08
BKZ 56 c	113897	0.00	0.13	0.00	4.86	0.00	3600.00	31.70	899.83	0.96	69.83
BKZ 57 c	116775	0.00	0.13	0.03	4.96	0.03	3600.00	31.20	76.12	1.29	74.18
BKZ 58 c	114538	0.00	0.13	0.00	4.85	0.00	3600.00	32.87	52.76	1.39	66.00
BKZ 59 c	114481	0.00	0.12	0.00	4.87	0.00	3600.00	35.50	106.76	1.05	72.73
BKZ 510 c	120370	0.00	0.13	0.00	4.86	0.00	3600.00	30.98	621.35	1.04	62.96
BKZ 61 c	82234	0.00	0.09	0.00	3.33	0.00	3600.00	7.99	9.22	2.15	29.91
BKZ 62 c	80523	0.00	0.09	0.01	3.33	0.01	3600.00	20.15	8.17	1.79	28.30
BKZ 63 c	83558	0.00	0.09	0.00	3.34	0.02	3600.00	17.48	12.82	3.15	23.39
BKZ 64 c	79162	0.00	0.09	0.00	3.35	0.00	3600.00	15.96	6.92	1.95	23.34
BKZ 65 c	75407	0.00	0.08	0.07	3.38	0.07	3600.00	25.92	8.68	3.81	32.43
Average		0.28	0.10	0.26	3.67	0.25	3600.00	22.18	107.42	2.20	40.80

Table B.15: Comparison among the proposed heuristics to the set BKZ for $T = 0.2 \times |M|$.

Instance	$\rho(X^{\text{B\&C}})$	AMU		SBA		PR R Any		LPH		PM	
		Dev (%)	T (s)	Dev (%)	T(s)	Dev (%)	T (s)	Dev (%)	T (s)	Dev (%)	T (s)
BKZ 41	84057	**0.00**	3.22	**0.00**	3.24	**0.00**	3600.00	10.72	13.88	2.83	26.89
BKZ 42	86949	**0.00**	3.22	**0.00**	3.20	**0.00**	3600.00	9.53	9.11	3.29	25.31
BKZ 43	81116	**0.00**	3.19	**0.00**	3.21	**0.00**	3600.00	13.06	6.89	4.91	23.75
BKZ 44	84919	**0.00**	3.24	**0.00**	3.22	**0.00**	3600.00	14.27	10.94	3.35	24.60
BKZ 45	88603	**0.00**	3.22	**0.00**	3.19	**0.00**	3600.00	14.09	15.92	5.94	25.75
BKZ 46	85543	**0.00**	3.21	**0.00**	3.23	**0.00**	3600.00	9.06	11.51	6.41	26.59
BKZ 47	83015	**0.00**	3.23	**0.00**	3.21	**0.00**	3600.00	16.14	13.70	4.72	24.43
BKZ 48	86638	**0.00**	3.20	**0.00**	3.19	**0.00**	3600.00	14.29	10.08	4.09	24.92
BKZ 49	80300	**0.00**	3.23	**0.00**	3.22	**0.00**	3600.00	10.76	13.57	4.96	27.02
BKZ 410	78350	**0.00**	3.22	**0.00**	3.18	**0.00**	3600.00	11.12	9.74	6.17	25.75
BKZ 51	146862	**0.00**	5.93	**0.00**	5.96	**0.00**	3600.00	4.25	161.70	1.40	72.16
BKZ 52	142741	**0.00**	5.95	**0.00**	5.95	**0.00**	3600.00	6.04	343.42	1.86	86.69
BKZ 53	145054	**0.00**	5.95	**0.00**	5.92	**0.00**	3600.00	3.87	58.78	1.76	80.40
BKZ 54	147638	**0.00**	5.98	**0.00**	6.03	**0.00**	3600.00	6.92	84.94	1.91	83.10
BKZ 55	146392	0.00	5.93	**-0.01**	5.87	**-0.01**	3600.00	6.32	409.44	1.65	88.92
BKZ 56	140342	**0.00**	5.97	**0.00**	5.98	**0.00**	3600.00	5.02	70.65	2.12	93.39
BKZ 57	148061	**0.00**	5.94	**0.00**	5.98	**0.00**	3600.00	5.28	108.38	2.78	86.45
BKZ 58	142878	**0.00**	5.98	**0.00**	5.95	**0.00**	3600.00	5.27	144.28	1.97	81.83
BKZ 59	140573	**0.00**	5.91	**0.00**	5.96	**0.00**	3600.00	5.09	45.86	2.13	90.37
BKZ 510	144963	**0.00**	5.99	**0.00**	5.95	**0.00**	3600.00	4.41	212.95	2.15	81.70
BKZ 61	80727	**0.00**	3.92	**0.00**	3.92	**0.00**	3600.00	12.32	5.20	6.05	33.86
BKZ 62	83328	**0.00**	3.99	**0.00**	3.94	**0.00**	3600.00	10.37	3.31	3.14	33.73
BKZ 63	83817	**0.00**	3.95	**0.00**	3.96	**0.00**	3600.00	9.10	6.51	5.48	31.05
BKZ 64	84911	**0.00**	3.93	**0.00**	3.94	**0.00**	3600.00	14.90	11.64	4.74	28.13
BKZ 65	76058	**0.00**	3.91	**0.00**	3.92	**0.00**	3600.00	17.98	9.67	3.72	34.15
BKZ 41 b	83109	**0.00**	3.21	**0.00**	3.24	**0.00**	3600.00	11.64	10.95	5.40	24.60
BKZ 42 b	83346	**0.00**	3.20	**0.00**	3.23	**0.00**	3600.00	10.07	14.07	5.50	25.66
BKZ 43 b	86101	**0.00**	3.22	**0.00**	3.22	**0.00**	3600.00	15.15	15.42	6.08	23.15
BKZ 44 b	84309	**0.00**	3.25	**0.00**	3.22	**0.00**	3600.00	10.06	14.05	4.65	25.81
BKZ 45 b	82577	**0.00**	3.21	**0.00**	3.17	**0.00**	3600.00	13.46	11.22	4.08	27.62
BKZ 46 b	75800	**0.00**	3.23	**0.00**	3.23	**0.00**	3600.00	14.01	7.77	4.71	29.14
BKZ 47 b	79865	**0.00**	3.22	**0.00**	3.22	**0.00**	3600.00	11.08	7.93	3.58	26.40
BKZ 48 b	91065	**0.00**	3.24	**0.00**	3.27	**0.00**	3600.00	8.50	9.10	2.35	23.59
BKZ 49 b	88552	**0.00**	3.26	**0.00**	3.24	**0.00**	3600.00	9.10	11.64	3.54	26.07
BKZ 410 b	79189	**0.00**	3.21	**0.00**	3.20	**0.00**	3600.00	13.44	3.86	6.01	29.34
BKZ 51 b	142540	**0.00**	5.97	**0.00**	5.99	**0.00**	3600.00	5.71	35.91	2.72	69.34
BKZ 52 b	141960	**0.00**	5.97	**0.00**	5.95	**0.00**	3600.00	4.37	74.61	2.01	85.42
BKZ 53 b	139088	**0.00**	5.92	**0.00**	5.97	**0.00**	3600.00	4.75	361.94	1.71	89.90
BKZ 54 b	141161	**0.00**	6.00	**0.00**	6.03	**0.00**	3600.00	4.25	76.12	1.87	85.37
BKZ 55 b	149953	**0.00**	5.96	**0.00**	5.95	**0.00**	3600.00	5.14	416.87	1.48	77.13
BKZ 56 b	146290	**0.00**	5.97	**0.00**	5.99	**0.00**	3600.00	5.07	54.81	1.98	82.45
BKZ 57 b	143834	**0.00**	6.00	**0.00**	5.94	**0.00**	3600.00	3.29	899.69	1.79	96.16
BKZ 58 b	150488	0.00	5.97	**-0.04**	5.97	**-0.04**	3600.00	5.72	196.99	1.99	76.52
BKZ 59 b	144310	**0.00**	5.97	**0.00**	5.90	**0.00**	3600.00	5.38	131.35	2.00	90.42
BKZ 510 b	145583	**0.00**	5.93	**0.00**	5.96	**0.00**	3600.00	4.00	545.91	1.61	93.70
BKZ 61 b	79698	**0.00**	3.95	**0.00**	3.92	**0.00**	3600.00	11.83	5.53	5.16	31.21
BKZ 62 b	77616	**0.00**	4.00	**0.00**	3.93	**0.00**	3600.00	11.21	3.93	4.43	31.80
BKZ 63 b	82638	**0.00**	3.99	**0.00**	3.96	**0.00**	3600.00	9.06	12.23	3.72	32.81
BKZ 64 b	86076	**0.00**	3.91	**0.00**	3.95	**0.00**	3600.00	10.75	5.59	4.25	35.50
BKZ 65 b	88496	**0.00**	3.92	**0.00**	3.96	**0.00**	3600.00	15.12	9.86	5.01	34.65
BKZ 41 c	81126	**0.00**	3.21	**0.00**	3.24	**0.00**	3600.00	9.92	10.43	3.92	26.94
BKZ 42 c	88322	**0.00**	3.23	**0.00**	3.18	**0.00**	3600.00	8.20	16.82	4.40	23.69
BKZ 43 c	86852	**0.00**	3.22	**0.00**	3.25	**0.00**	3600.00	7.73	26.40	3.35	26.64
BKZ 44 c	86673	**0.00**	3.21	**0.00**	3.27	**0.00**	3600.00	9.33	9.06	4.44	26.13
BKZ 45 c	86569	**0.00**	3.19	**0.00**	3.22	**0.00**	3600.00	11.15	10.27	4.28	27.10
BKZ 46 c	77894	**0.00**	3.23	**0.00**	3.22	**0.00**	3600.00	15.74	9.51	5.60	26.50
BKZ 47 c	82091	**0.00**	3.19	**0.00**	3.19	**0.00**	3600.00	12.44	7.43	5.97	20.11
BKZ 48 c	83589	**0.00**	3.21	**0.00**	3.23	**0.00**	3600.00	14.08	12.50	4.93	27.80
BKZ 49 c	88519	**0.00**	3.21	**0.00**	3.17	**0.00**	3600.00	9.94	8.05	3.68	25.59
BKZ 410 c	82546	**0.00**	3.21	**0.00**	3.19	**0.00**	3600.00	12.61	6.01	4.74	24.04
BKZ 51 c	146996	**0.00**	5.96	**0.00**	5.95	**0.00**	3600.00	5.37	208.71	1.82	80.72
BKZ 52 c	140764	0.00	5.93	**-0.01**	5.93	**-0.01**	3600.00	6.23	100.97	2.72	81.12
BKZ 53 c	145415	**0.00**	5.90	**0.00**	5.94	**0.00**	3600.00	5.66	85.37	1.69	80.67
BKZ 54 c	148148	**0.00**	5.95	**0.00**	6.03	**0.00**	3600.00	3.46	899.55	1.71	89.51
BKZ 55 c	139314	**0.00**	5.91	**0.00**	5.92	**0.00**	3600.00	5.50	112.77	1.82	82.69
BKZ 56 c	142511	**0.00**	5.95	**0.00**	5.95	**0.00**	3600.00	5.25	899.83	1.97	89.00
BKZ 57 c	143687	**0.00**	5.96	**0.00**	5.94	**0.00**	3600.00	6.63	76.12	2.12	87.82
BKZ 58 c	143999	**0.00**	5.91	**0.00**	5.95	**0.00**	3600.00	5.68	52.76	2.36	86.27
BKZ 59 c	145520	0.00	6.00	**-0.01**	5.97	**-0.01**	3600.00	6.60	106.76	2.23	84.69
BKZ 510 c	150995	**0.00**	5.96	**0.00**	5.90	**0.00**	3600.00	4.41	621.35	1.54	85.30
BKZ 61 c	81151	**0.00**	3.93	**0.00**	3.92	**0.00**	3600.00	9.43	9.22	3.49	36.23
BKZ 62 c	85719	**0.00**	3.98	**0.00**	3.98	**0.00**	3600.00	12.86	8.17	4.94	31.09
BKZ 63 c	86575	**0.00**	3.94	**0.00**	4.03	**0.00**	3600.00	13.38	12.82	4.15	30.93
BKZ 64 c	80752	**0.00**	3.92	**0.00**	3.96	**0.00**	3600.00	13.68	6.92	4.19	32.69
BKZ 65 c	81370	**0.00**	3.95	**0.00**	3.91	**0.00**	3600.00	16.69	8.68	5.44	36.61
Average		**0.00**	4.46	**0.00**	4.46	**0.00**	3600.00	9.26	107.41	3.53	50.73

Table B.16: Comparison among the proposed heuristics to the set BKZ for $T = 0.3 \times |M|$.

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Amadeu ALMEIDA COCO

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Problème de couverture robuste : formulations, algorithmes et une application

Deux problèmes robustes d'optimisation NP-difficiles sont étudiés dans cette thèse: le problème min-max regret de couverture pondérée (min-max regret WSCP) et le problème min-max regret de couverture et localisation maximale (min-max regret MCLP). Les données incertaines dans ces problèmes sont modélisées par des intervalles et seules les valeurs minimales et maximales pour chaque intervalle sont connues. Le min-max regret WSCP a été investigué notamment dans le cadre théorique, alors que le min-max regret MCLP a des applications en logistique des catastrophes étudiées dans cette thèse. Deux autres critères d'optimisation robuste ont été dérivés pour le MCLP: le max-max MCLP et le min-max MCLP. En matière de méthodes, formulations mathématiques, algorithmes exacts et heuristiques ont été développés et appliqués aux deux problèmes. Des expérimentations computationnelles ont montré que les algorithmes exacts ont permis de résoudre efficacement 14 des 75 instances générées par le min-max regret WSCP et toutes les instances réalistes pour le min-max regret MCLP. Pour les cas simulés qui n'ont pas été résolus de manière optimale dans les deux problèmes, les heuristiques développées dans cette thèse ont trouvé des solutions aussi bien ou mieux que le meilleur algorithme exact dans presque tous les cas. En ce qui concerne l'application en logistique des catastrophes, les modèles robustes ont trouvé des solutions similaires pour les scénarios réalistes des tremblements de terre qui a eu lieu à Katmandu au Népal en 2015. Cela indique que nous avons une solution robuste.

Mots clés : optimisation combinatoire - métaheuristiques - algorithmes - recherche - logistique.

Robust Covering Problems: Formulations, Algorithms and an Application

Two robust optimization NP-Hard problems are studied in this thesis: the min-max regret Weighted Set Covering Problem (min-max regret WSCP) and the min-max regret Maximal Coverage Location Problem (min-max regret MCLP). The min-max regret WSCP and min-max regret MCLP are, respectively, the robust optimization counterparts of the Set Covering Problem and of the Maximal Coverage Location Problem. The uncertain data in these problems is modeled by intervals and only the minimum and maximum values for each interval are known. However, while the min-max regret WSCP is mainly studied theoretically, the min-max regret MCLP has an application in disaster logistics which is also investigated in this thesis. Two other robust optimization criteria were derived for the MCLP: the max-max MCLP and the min-max MCLP. In terms of methods, mathematical formulations, exact algorithms and heuristics were developed and applied to both problems. Computational experiments showed that the exact algorithms efficiently solved 14 out of 75 instances generated to the min-max regret WSCP and all realistic instances created to the min-max regret MCLP. For the simulated instances that was not solved to optimally in both problems, the heuristics developed in this thesis found solutions, as good as, or better than the exact algorithms in almost all instances. Concerning the application in disaster logistics, the robust models found similar solutions for realistic scenarios of the earthquakes that hit Kathmandu, Nepal in 2015. This indicates that we have got a robust solution, according to all optimization models.

Keywords: combinatorial optimization - metaheuristics - algorithms - operations research - logistics.

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