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Lab Computer Vision : Kalman Filter

The objective of this lab is to become familiar with linear stochastic probabilistic filters, such as linear Kalman filter.

Problem 1

Write a Matlab function that implements the linear discrete Kalman filter algorithm which takes as its inputs, the measure vector y, the number of measures N, the errors var_e=R and var_w=Q and which returns the resulted state vector S and the final error matrix P.
Test your algorithm for a one dimentional case where A=1, B=0, C=1. Initialize your system to :
P=1
X=sqrt(P)*0.8

Test your algorithm for N=10, 50, 100. What is your conclusion w.r.t to the sample size.

Problem 2 : Switching from the continous to the discrete KF

Let a target displacing on a plane for what we could measure the position $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ with some uncertainty. Consider that the target mouves with a constant unknown velocity $\begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix}$ meaning in theory that accelerations $\ddot{x}(t) = \ddot{y}(t) = 0$. In practice, there is a small variations in the acceleration such as : $\begin{matrix} \ddot{x}(t) = w_x(t) \\ \ddot{y}(t) = w_y(t) \end{matrix}$ where $w(t) = \begin{pmatrix} w_x(t) \\ w_y(t) \end{pmatrix}$ is a random vector of zero mean and variance-covariance matrix defined by :

$$E\left(w(t)w^T(t)\right) = \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_y \end{pmatrix} = Q_c$$

1) Show that such a modelisation admits the following state representation :

$$\dot{X}(t) = AX(t) + Bw(t)$$

Where $X(t) = \begin{pmatrix} x(t) & \dot{x}(t) & y(t) & \dot{x}(t) \end{pmatrix}$ with A, B two matrices to be precised.

To complete the model, we consider the following observation equation :

$$Z(t) = CX(t) + b(t)$$

Where C is the measurment matrix to be precised and $b(t)$ the random measure noise of zero mean and variance-covariance matrix:

$$E\left(b(t)b^T(t)\right) = \begin{pmatrix} \theta_x & 0 \\ 0 & \theta_y \end{pmatrix}$$

2) The classical solution of the state equation is :

$$X(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-u)}Bw(u)du \quad t > t_0$$

That we descritize by puting $t_0 = nT \quad and \quad t = T + nT$:

$$\begin{aligned} X((n+1)T) &= e^{AT}X(nT) + w(n) \\ w(n) &= \int_{nT}^{(n+1)T} e^{A(nT+T-u)}Bw(u)du \end{aligned}$$

Using the matrix A found above and linearizing by the taylor devlopment, show that

$$e^{At} = \begin{pmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The last step of the discretisation is to find the variance-covariance matrix of w(n). Using the fact that w(t) is a white noise such that $E\left(w(u)w(v)\right) = Q_c\delta(u-v)$ show that :

$$Q(n) = E\left(w(n)w^T(n)\right) = \int_0^T e^{Av}BQ_cB^Te^{A^Tv}dv$$

and that by the Taylor devlopment that :

$$Q(n) = \begin{pmatrix} \sigma_x^2\frac{T^3}{3} & \sigma_x^2\frac{T^2}{2} & 0 & 0 \\ \sigma_x^2\frac{T^2}{2} & \sigma_x^2 T & 0 & 0 \\ 0 & 0 & \sigma_y^2\frac{T^3}{3} & \sigma_y^2\frac{T^2}{2} \\ 0 & 0 & \sigma_y^2\frac{T^2}{2} & \sigma_y^2 T \end{pmatrix}$$

Similarly, it is easy to show that

$$Z(n) = CX(n) + b(n) \quad with$$

$$R = \begin{pmatrix} \frac{\sigma_x^2}{T} & 0 \\ 0 & \frac{\sigma_y^2}{T} \end{pmatrix}$$

R is the variance-covariance matrix of b(t).
Write the final discret state equation model of the target.

Problem 3 : Simulation

We aim by this problem to estimate the angular velocity, the angular position (tilt) and the bias given respectively by the gyroscope and the accelerometer of an inertial unit.

The accelerometer role is to measure the tilt α_M obtained by adding the accelorometer noise b_2 and the real value of the mobile tilt α

$$\alpha_M = \alpha + b_2$$

The gyroscope measures the velocity u of the mobile, obtained by adding the gyroscope noise b_1 and the gyroscope bias b and the real angular velocity of the mobile $\dot{\alpha}$

$$u = \dot{\alpha} + b + b_1$$

Assume that there is no acceleration, the parameters evolution could be modelled as follows :

$$\begin{cases} X_{k+1} = AX_k + w_k \\ Y_k = CX_k + v_k \end{cases}$$

Where

$$X_k = \begin{pmatrix} \dot{\alpha} \\ \alpha \\ b \end{pmatrix}_k, \quad Y = \begin{pmatrix} u \\ \alpha_M \end{pmatrix} \quad A = \begin{pmatrix} 1 & 0 & 0 \\ t_e & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$v = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad E(vv^T) = R = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}, \quad E(ww^T) = Q = \begin{pmatrix} \varepsilon_{\dot{\alpha}} & 0 & 0 \\ 0 & \varepsilon_\alpha & 0 \\ 0 & 0 & \varepsilon_b \end{pmatrix}$$

Hypothesis : Assume that
1- The sampling frequency fe (=1/Te) is equal to 50.
2- The ground truth (real measures without noise) are :

$$\dot{\alpha} = 2\pi^2 \sin(2\pi t), \quad \alpha = \pi - \pi\cos(2\pi t), \quad b = 10t + 20\sin(0.2\pi t)$$

Where t is the measurement time in second wich is put to 10 here
3-The sensors noises are assumed to be : $\sigma_1 = 0.6\pi^2, \quad \sigma_2 = 0.2\pi$
4- In order to simulate a real noised measures, add a randn noise (multiplied by the respective sensor noises) to the ground truth.

5-Initialisation :

$A, C, R \ (given).\ Diag(Q) = \begin{bmatrix} 10 & 0 & 2 \end{bmatrix}$

$P = zeros(3,3),\ X(1) = 0, X(2) = 0, X(3) = 0$

$X_{estim} = zeros(temp * f_e)$

$X_{estim}(1,i) = X(i)_{i=1:3}$

- ✓ P is symmetric positive definite matrix. The final estimated errors on each state parameter is the diagonal of P.
- ✓ Use the Kalman filter to estimate the three unknown parameters : the angular velocity, the tilt and the bias of the gyroscope.
- ✓ Plot respectively the ground truth, the measures and the estimated state for each of the three unknowns.